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Original article

Explicit difference method for solving the two-dimensional two-sided fractional diffusion equation in the shifted grunwald estimate form

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In this paper, we introduce and discuss an algorithm for the numerical solution of two-dimensional two-sided fractional diffusion equation. The algorithm for the numerical solution of this equation is based on explicit finite difference approximation. Consistency, conditional stability, and convergence of the fractional order numerical method are described. Finally, numerical example is provided to show that the numerical method for solving this equation is an effective solution method.

1. Introduction

Fractional calculus is becoming a useful and, in some cases, key tool in the analysis of scientific problems in abroad array of fields such as physics, engineering, biology, and economics. In particular, fractional partial differential equations have turned out to be especially relevant. For example, fractional diffusion equations have been successfully used to describe diffusion processes where the diffusion is anomalous (Miller and Ross, 1993; Meerschaert and Tadjeran, 2004, 2006, 2007; Podlubny, 1999; Roop, 2005; Lin and Liu, 2004; Meerschaert et al., 2006; Liu, 2009; Joaquín and Santos, 2011), and fractional diffusion.

Difference methods and, in particular, explicit difference methods, are an important class of numerical methods for solving fractional and normal differential equations. The usefulness of the explicit methods and there as on why they are widely employed is based on their particularly attractive features (Yuste and Acedo, 2005;

Morton and Mayers, 1994): flexibility, simplicity, scanty computational demand, and the possibility of easy generalization to spatial dimensions higher than1.

The method discussed in this paper is an explicit finite difference method designed for solving the two-dimensional two-sided fractional diffusion where the fractional derivative is in the shifted Grunwald estimate form. The conditional stability and convergence of the explicit finite difference approximation are analyzed and finally, we will present example to show the efficiency of our numerical method.

2. Explicit difference method for solving the two-dimensional two-sided fractional diffusion equation

In this section, we use the explicit finite difference method for solving the two-dimensional two-sided fractional diffusion equation of the form:

$$\frac{\partial u(x,y,t)}{\partial t} = a(x,y)\left[(1-d)\frac{\partial^{\alpha} u(x,y,t)}{\partial_{-} x^{\alpha}} + d\,\frac{\partial^{\alpha} u(x,y,t)}{\partial_{+} x^{\alpha}}\right] + b(x,y)\left[(1-e)\frac{\partial^{\beta} u(x,y,t)}{\partial_{-} y^{\beta}} + e\frac{\partial^{\beta} u(x,y,t)}{\partial_{+} y^{\beta}}\right] + q(x,y,t) \qquad (1)$$

In this problem initial and boundary conditions are considered which are:

u(x,y,0)= φ (x,y), for x0 < x < xR and y0 < y < yR
u (x0,y,t) = 0, for y0 < y< yR and 0≤ t ≤ T
u(x,y0,t) = 0, for x0 < x < xR and 0≤ t ≤ T
u (xR,y, t) = ψ1 (y,t), for y0 < y< yR and 0≤ t ≤ T
u (x,yR,t) = ψ2 (x,t), for x0 < x < xR and 0≤ t ≤ T

where a, b, and φ are known functions of x and y , and the weights $d, e, 1-d, 1-e \in [0,1]$. ψ1 is a known function of y and t, ψ2 is a known function of x and t. α and β are given fractional number. q is a knwon function of x, y and t.

The left–handed $\partial^{\alpha} u / \partial_{-} x^{\alpha}$, $\partial^{\beta} u / \partial_{-} y^{\beta}$ and the right–handed $\partial^{\alpha} u / \partial_{+} x^{\alpha}$, $\partial^{\beta} u / \partial_{+} y^{\beta}$ fractional derivatives by the shifted Grunwald estimate formulae are [2, 7]:

$$\frac{\partial^{\alpha} u(x,y,t)}{\partial_{+} x^{\alpha}} = \frac{1}{(\Delta x)^{\alpha}} \sum_{k=0}^{i+1} g_{\alpha,k} u^{s}_{i-k+1,j,f} + O(\Delta x)$$

$$\frac{\partial^{\beta} u(x,y,t)}{\partial_{+} y^{\beta}} = \frac{1}{(\Delta y)^{\beta}} \sum_{k=0}^{j+1} g_{\beta,k} u^{s}_{i,j-k+1,f} + O(\Delta y)$$

(2)

$$\frac{\partial^{\alpha} u(x,y,t)}{\partial_{-} x^{\alpha}} = \frac{1}{(\Delta x)^{\alpha}} \sum_{k=0}^{n-i+1} g_{\alpha,k} u^{s}_{i+k-1,j,f} + O(\Delta x)$$

$$\frac{\partial^{\beta} u(x,y,t)}{\partial_{-} y^{\beta}} = \frac{1}{(\Delta y)^{\beta}} \sum_{k=0}^{m-j+1} g_{\beta,k} u^{s}_{i,j+k-1,f} + O(\Delta y)$$

The finite difference method starts by dividing the x-interval [x0, xR] into n subintervals to get the grid points xi = x0 + iΔx, where $\Delta x = (x_R - x_0)/n$ and i = 0,1,…,n. and we divide the y-interval [y0, yR] into m subintervals to get the grid points yj = y0+jΔy, where $\Delta y = (y_R - y_0)/m$ and j=0,1,…,m.

Also, the t-interval [0,T] is divided into M subintervals to get the grid points ts = sΔt, s = 0,…, M, where $\Delta t = T/M$.

Now, we evaluate eq. (1) at (xi , y j ,t s) and use the explicit Euler method to get

$$\frac{u(x_i, y_j, t_{s+1}) - u(x_i, y_j, t_s)}{\Delta t} = a(x_i, y_j)\left[(1-d)\frac{\partial^\alpha u(x_i, y_j, t_s)}{\partial_- x^\alpha} + d\frac{\partial^\alpha u(x_i, y_j, t_s)}{\partial_+ x^\alpha}\right] +$$

$$b(x_i, y_j)\left[(1-e)\frac{\partial^\beta u(x_i, y_j, t_s)}{\partial_- y^\beta} + e\frac{\partial^\beta u(x_i, y_j, t_s)}{\partial_+ y^\beta}\right] +$$

$$q(x_i, y_j, t_s) \qquad (3)$$

Use fractional derivative of the shifted Grunwald estimate eq.(2), to reduce eq.(3) to the following form:

$$\frac{u_{i,j}^{s+1} - u_{i,j}^{s}}{\Delta t} = a(x_i, y_j)\left[(1-d)\frac{1}{\Delta x^\alpha}\sum_{k=0}^{n-i+1} g_{\alpha,k} u_{i+k-1,j}^{s} + d\frac{1}{\Delta x^\alpha}\sum_{k=0}^{i+1} g_{\alpha,k} u_{i-k+1,j}^{s}\right] +$$

$$b(x_i, y_j)\left[(1-e)\frac{1}{\Delta y^\beta}\sum_{k=0}^{m-j+1} g_{\beta,k} u_{i,j+k-1}^{s} + e\frac{1}{\Delta y^\beta}\sum_{k=0}^{j+1} g_{\beta,k} u_{i,j-k+1}^{s}\right] +$$

$$q(x_i, y_j, t_s)$$

$$\frac{u_{i,j}^{s+1} - u_{i,j}^{s}}{\Delta t} = (1-d)\frac{a_{i,j}}{\Delta x^\alpha}\sum_{k=0}^{n-i+1} g_{\alpha,k} u_{i+k-1,j}^{s} + d\frac{a_{i,j}}{\Delta x^\alpha}\sum_{k=0}^{i+1} g_{\alpha,k} u_{i-k+1,j}^{s} +$$

$$(1-e)\frac{b_{i,j}}{\Delta y^\beta}\sum_{k=0}^{m-j+1} g_{\beta,k} u_{i,j+k-1}^{s} + e\frac{b_{i,j}}{\Delta y^\beta}\sum_{k=0}^{j+1} g_{\beta,k} u_{i,j-k+1}^{s} +$$

$$q_{i,j}^{s}$$

$$i = 1,\ldots,n-1,\; j = 1,\ldots,m-1,\; s = 0,\ldots,M \qquad (4)$$

Where $u_{i,j}^{s} = u(x_i, y_j, t_s)$, $a_{i,j} = a(x_i, y_j)$, $b_{i,j,f} = b(x_i, y_j, z_f)$, $c_{i,j} = c(x_i, y_j)$,

$q_{i,j}^{s} = q(x_i, y_j, t_s)$, $g_{\alpha,k} = (-1)^k \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$, k=0,1,2,… and

$$g_{\beta,k} = (-1)^k \frac{\beta(\beta-1)\cdots(\beta-k+1)}{k!}$$, k=0,1,2,…

The resulting equation can be explicitly solved for $u_{i,j}^{s+1}$ to give

$$u_{i,j}^{s+1} = (1-d)a_{i,j}\frac{\Delta t}{\Delta x^{\alpha}}\sum_{k=0}^{n-i+1} g_{\alpha,k}u_{i-k+1,j}^{s} + da_{i,j}\frac{\Delta t}{\Delta x^{\alpha}}\sum_{k=0}^{i+1} g_{\alpha,k}u_{i-k+1,j}^{s} +$$

$$(1-e)b_{i,j}\frac{\Delta t}{\Delta y^{\beta}}\sum_{k=0}^{m-j+1} g_{\beta,k}u_{i,j+k-1}^{s} + eb_{i,j}\frac{\Delta t}{\Delta y^{\beta}}\sum_{k=0}^{j+1} g_{\beta,k}u_{i,j-k+1}^{s} +$$

$$u_{i,j}^{s} + \Delta t q_{i,j}^{s} \qquad (5)$$

Also from the initial condition and boundary conditions one can get

$u_{i,j}^{0} = \varphi_{i,j}$, i=0,…, n and j=0,…, m .

$u_{0,j}^{s} = 0$, j=0,…, m and s=1,…,M

$u_{i,0}^{s} = 0$, i=0,…, n and s=1,…,M

$u_{R,j}^{s} = \psi_{j}^{s}$, j=0,…, m and s=1,…,M

$u_{i,R}^{s} = \psi_{i}^{s}$, i=0,…, n and s=1,…,M

Where $\varphi_{i,j} = \varphi(x_i, y_j)$, $\psi_j^s = \psi(y_j, t_s)$ and $\psi_i^s = \psi(x_i, t_s)$

3. Stability of explicit difference method two-dimensional two-sided fractional diffusion equation

We define the following fractional partial difference operator:

$$\omega_{\alpha,x}u_{i,j}^{s} = (1-d)a_{i,j}\frac{\Delta t}{\Delta x^{\alpha}}\sum_{k=0}^{n-i+1} g_{\alpha,k}u_{i+k-1,j}^{s} + da_{i,j}\frac{\Delta t}{\Delta x^{\alpha}}\sum_{k=0}^{i+1} g_{\alpha,k}u_{i-k+1,j}^{s}$$

which is of $O(\Delta x)$ approximation to the α th fractional derivative. Similarly, the following fractional partial difference operator is defined.

$$\omega_{\beta,y}u_{i,j}^{s} = (1-e)b_{i,j}\frac{\Delta t}{\Delta y^{\beta}}\sum_{k=0}^{m-j+1} g_{\beta,k}u_{i,j+k-1}^{s} + eb_{i,j}\frac{\Delta t}{\Delta y^{\beta}}\sum_{k=0}^{j+1} g_{\beta,k}u_{i,j-k+1}^{s}$$

is of $O(\Delta y)$ approximation of the β -order Grunwald shifted fractional derivative term.

With these definitions, the explicit difference scheme (5) may be written in the following compact form:

$$u_{i,j}^{s+1} = (1 + \Delta t\omega_{\alpha,x} + \Delta t\omega_{\beta,y})u_{i,j}^{s} + q_{i,j}^{s}\Delta t \qquad (6)$$

The above three-dimensional two-sided of explicit method has local truncation error of the form

$$O(\Delta t) + O(\Delta x) + O(\Delta y).$$

eq.(6) may be written in form

$$u_{i,j}^{s+1} = (1 + \Delta t\omega_{\alpha,x})(1 + \Delta t\omega_{\beta,y})u_{i,j}^{s} + \Delta t q_{i,j}^{s} \qquad (7)$$

which introduces an additional perturbation error equal to

$$(\Delta t)^2 \omega_{\beta,x}\omega_{\alpha,y}u_{i,j}^s$$

Then

$$\underline{U}^{s+1} = \underline{ETU}^s + \underline{R}^s \tag{8}$$

where

$$E = (1 + \Delta t \omega_{\alpha,x}),$$

$$T = (1 + \Delta t \omega_{\beta,y}),$$

and

$$\underline{U}^s = [u_{1,1}^s, u_{2,1}^s, \ldots, u_{n-1,1}^s, u_{1,2}^s, u_{2,2}^s, \ldots, u_{n-1,2}^s, \ldots, u_{1,m-1}^s, u_{2,m-1}^s, \ldots, u_{n-1,m-1}^s]^T$$

and the vector $\underline{R}^s$ is the forcing term .

To solve the problem for each fixed yj to obtain an intermediate solution $u_{i,j}^{s/2}$ from

$$u_{i,j}^{s+1} = (1 + \Delta t \omega_{\alpha,x})u_{i,j}^{s/2} + \Delta t q_{i,j}^s \tag{9}$$

Then solve for each fixed xi

$$u_{i,j,f}^{s/2} = (1 + \Delta t \omega_{\beta,y})u_{i,j}^s \tag{10}$$

Now, we must prove that each one-dimensional explicit system defined by the linear difference eqs. (9) and (10) is conditionally stable for all 1 < α , β < 2.

Theorem: The explicit system defined by the linear difference eqs.(9) and (10) with 1 < α , β < 2 is conditionally stable if

$$\frac{\Delta t}{\Delta x^\alpha} \le \frac{1}{\alpha\left[(1-a)d_{\max} + ad_{\max}\right]}$$

and

$$\frac{\Delta t}{\Delta y^\beta} \le \frac{1}{\beta\left[(1-b)e_{\max} + be_{\max}\right]}$$

Proof:

At each grid point yk, for $k = 1, \ldots, m-1$, the system of equations defined by eq.(10) can be written in the explicit matrix form $\underline{U_k^{s+1}} = \underline{C_k}\underline{U_k^{s/2}} + \Delta t \underline{Q_k^s}$ where

$$\underline{U_k^{s+1}} = [u_{1,k}^{s+1}, u_{2,k}^{s+1}, \ldots, u_{n-1,k}^{s+1}]^T,$$

$$\underline{U_k^{s/2}} = [u_{1,k}^{s/2}, u_{2,k}^{s/2}, \ldots, u_{n-1,k}^{s/2}]^T,$$

$$\Delta t \underline{Q_k^s} = [q_{1,k}^s \Delta t, q_{2,k}^s \Delta t, \ldots, q_{n-1,k}^s \Delta t]^T$$

Therefore the resulting matrix entries $C_{i,j}$ for $i = 1, \ldots, n-1$ and $j = 1, \ldots, n-1$ are defined by

$$C_{i,j} = \begin{cases} 1+\xi_{i,k} g_{\alpha,1} + \eta_{i,k} g_{\alpha,1} & for \quad j=i \\ \xi_{i,k} g_{\alpha,0} + \eta_{i,k} g_{\alpha,2} & for \quad j=i-1 \\ \xi_{i,k} g_{\alpha,2} + \eta_{i,k} g_{\alpha,0} & for \quad j=i+1 \\ \eta_{i,k} g_{\alpha,j+1} & for \quad j<i+1 \\ \xi_{i,k} g_{\alpha,i-j+1} & for \quad j>i+1 \end{cases}$$

To illustrate this matrix pattern, we list the corresponding equations for the rows i =1, 2 and n-1:

$$u_{1,k}^{s+1} = (\xi_{1,k} g_{\alpha,0} + \eta_{1,k} g_{\alpha,2}) u_{0,k}^{s/2} + (1+\xi_{1,k} g_{\alpha,1} + \eta_{1,k} g_{\alpha,1}) u_{1,k}^{s/2} + (\xi_{1,k} g_{\alpha,2} + \eta_{1,k} g_{\alpha,0}) u_{2,k}^{s/2} + \xi_{1,k} g_{\alpha,3} u_{3,k}^{s/2} + \cdots + \xi_{1,k} g_{\alpha,k} u_{K,k}^{s/2} + \Delta t q_{1,k}^{s}$$

for i = 2 we have

$$u_{2,k}^{s+1} = \eta_{2,k} g_{\alpha,3} u_{0,k}^{s/2} + (\xi_{2,k} g_{\alpha,0} + \eta_{2,k} g_{\alpha,2}) u_{1,k}^{s/2} + (1+\xi_{2,k} g_{\alpha,1} + \eta_{2,k} g_{\alpha,1}) u_{2,k}^{s/2} + (\xi_{2,k} g_{\alpha,2} + \eta_{2,k} g_{\alpha,0}) u_{3,k}^{s/2} + \cdots + \xi_{2,k} g_{\alpha,k} u_{K-1,k}^{s/2} + \Delta t q_{2,k}^{s}$$

and for i = n - 1 we get

$$u_{n-1,k}^{s+1} = \eta_{n-1,k} g_{\alpha,n} u_{0,k}^{s/2} + \cdots + (\xi_{n-1,k} g_{\alpha,0} + \eta_{n-1,k} g_{\alpha,2}) u_{n-2,k}^{s/2} + (1+\xi_{n-1,k} g_{\alpha,1} + \eta_{n-1,k} g_{\alpha,1}) u_{n-1,k}^{s/2} + (\xi_{n-1,k} g_{\alpha,2} + \eta_{n-1,k} g_{\alpha,0}) u_{n,k}^{s/2} + \cdots + \xi_{n-1,k} g_{\alpha,k-n+2} u_{K,k}^{s/2} + \Delta t q_{n-1,k}^{s}$$

Where the coefficients

$\xi_{i,k} = (1-a) d_{i,j} \dfrac{\Delta t}{\Delta x^{\alpha}}$ and $\eta_{i,k} = a d_{i,j} \dfrac{\Delta t}{\Delta x^{\alpha}}$,

According to the Greshgorin theorem [9], the eigenvalues of the matrix C lie in the union of the circles centered at $c_{i,i}$ with radius

$$r_i = \sum_{\substack{l=0 \\ l\neq i}}^{n} c_{i,l}$$

Here we have $c_{i,i} = 1+\xi_{i,k} g_{\alpha,1} + \eta_{i,k} g_{\alpha,1} = 1-(\xi_{i,k}+\eta_{i,k})\alpha$ and

$$r_i = \sum_{\substack{l=0 \\ l\neq i}}^{n} c_{i,l} = \xi_{i,k} \sum_{\substack{l=0 \\ l\neq i}}^{n-i+1} g_{\alpha,i+l-1} + \eta_{i,k} \sum_{\substack{l=0 \\ l\neq i}}^{i+1} g_{\alpha,i-l+1} \le \xi_{i,k}\alpha + \eta_{i,k}\alpha$$

and therefore $c_{i,i} + r_i \le 1$. We also have

$$c_{i,i} - r_i \ge 1-(\xi_{i,k}+\eta_{i,k})\alpha - (\xi_{i,k}+\eta_{i,k})\alpha = 1-2(\xi_{i,k}+\eta_{i,k})\alpha$$

$$= 1-2\left[(1-a)d_{i,k}\frac{\Delta t}{\Delta x^{\alpha}} + a d_{i,k}\frac{\Delta t}{\Delta x^{\alpha}})\right]\alpha \ge 1-2\left[(1-a)d_{\max}\frac{\Delta t}{\Delta x^{\alpha}} + a d_{\max}\frac{\Delta t}{\Delta x^{\alpha}})\right]\alpha$$

Therefore, for the spectral radius of the matrix C to be at most one, it suffices to have

$$1-2\left[(1-a)d_{\max}\frac{\Delta t}{\Delta x^{\alpha}} + a d_{\max}\frac{\Delta t}{\Delta x^{\alpha}}\right]\alpha \ge -1 \rightarrow \left[(1-a)d_{\max}\frac{\Delta t}{\Delta x^{\alpha}} + a d_{\max}\frac{\Delta t}{\Delta x^{\alpha}}\right]\alpha \le 1$$

$$\left[(1-a)d_{\max}\alpha + a d_{\max}\alpha\right]\frac{\Delta t}{\Delta x^{\alpha}} \le 1 \rightarrow \qquad \frac{\Delta t}{\Delta x^{\alpha}} \le \frac{1}{\alpha\left[(1-a)d_{\max} + a d_{\max}\right]}$$

Same method above, resulting the system of equations defined by (10) is then defined by

$$\underline{S_k}\,\underline{U_k^s} = \underline{U_k^{s/2}},$$

where

$$\underline{U_k^s} = [u_{k,1}^s, u_{k,2}^s, \ldots, u_{k,m-1}^s]^T,$$
$$\underline{U_k^{s/2}} = [u_{k,1}^{s/2}, u_{k,2}^{s/2}, \ldots, u_{k,m-1}^{s/2}]^T,$$

$\underline{S_k}$ is the matrix of coefficients, and is the sum of a lower triangular matrix and a super diagonal matrix at the grid point xk for $k = 1, \ldots, n-1$. Therefore the resulting matrix entries $\underline{S_k}$ for $i = 1, 2, \ldots, m-1$ and $j = 1, \ldots, m-1$ are defined by

$$S_{i,j} = \begin{cases} 1 + \varsigma_{k,i} g_{\beta,1} + \psi_{k,i} g_{\beta,1} & for \quad j = i \\ \varsigma_{k,i} g_{\beta,0} + \psi_{k,i} g_{\beta,2} & for \quad j = i-1 \\ \varsigma_{k,i} g_{\beta,2} + \psi_{k,i} g_{\beta,0} & for \quad j = i+1 \\ \psi_{k,i} g_{\beta,j+1} & for \quad j < i+1 \\ \varsigma_{k,i} g_{\beta,i-j+1} & for \quad j > i+1 \end{cases}$$

Where the coefficients

$\varsigma_{i,k} = (1-b)e_{i,j}\dfrac{\Delta t}{\Delta y^{\beta}}$ and $\psi_{i,k} = be_{i,j}\dfrac{\Delta t}{\Delta y^{\beta}}$

So, and in the same way, According to the Greshgorin theorem [9], to get

$$\frac{\Delta t}{\Delta y^{\beta}} \le \frac{1}{\beta\left[(1-b)e_{\max} + be_{\max}\right]}$$

4. Consistency and convergent of explicit difference method two-dimensional two-sided fractional diffusion equation

To obtain the consistency of the two-dimensional two-sided fractional Diffusion equation, note that the time difference operator in (6) has a local truncation error of order $O(\Delta t),$ and the three space difference operators in (6) have local truncation errors of orders $O(\Delta x)$ and $O(\Delta y)$. Similar to Lemma 2.1 in paper of Meerschaert et al., (2006), we can obtain the following result:

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}\frac{\partial^{\beta}}{\partial y^{\beta}} f(x,y) = \omega_x^{\alpha}\omega_y^{\beta} f(x,y) + \mathrm{O}(\Delta x + \Delta y)$$

Which leads to the two-dimensional two-sided fractional Diffusion equation with order $O(\Delta t) + O(\Delta x) + O(\Delta y) + O(\Delta z)$.

We show above that explicit Euler method is consistent and conditionally stable, then by Laxs equivalence theorem,[12], it converges at the rate $O(\Delta x + \Delta y + \Delta z + \Delta t)$.

5. Numerical simulation and comparison

In this section, we implement the proposed method to solve two-dimensional two-sided fractional diffusion equation (1). Also, a comparison with numerical solution and exact solution, which is based on the explicit finite difference approximation of fractional derivative, is given.

Example: Consider the two-dimensional two-sided the fractional diffusion equation:

$$\frac{\partial u(x,y,t)}{\partial t}=\frac{\Gamma(1.4)(1-x)^{0.4}}{2}\left[0.4\frac{\partial^{1.6}u(x,y,t)}{\partial_+ x^{1.6}}+0.6\frac{\partial^{1.6}u(x,y,t)}{\partial_- x^{1.6}}\right]+\Gamma(0.2)(1-y)^{0.8}\left[0.7\frac{\partial^{1.8}u(x,y,t)}{\partial_+ y^{1.8}}+0.3\frac{\partial^{1.8}u(x,y,t)}{\partial_- y^{1.8}}\right]$$
$$-0.4x^{0.4}(1-x)^{0.4}ye^{-3t}-0.6ye^{-3t}-0.7x^2y^{-0.8}(1-y)^{0.8}e^{-3t}-0.3x^2e^{-3t}-3x^2ye^{-3t}$$

subject to the initial condition

u (x,y,0) = x2y, 0 < x < 1, 0 < y < 1

and the boundary conditions

u (0,y,t) = 0, 0 < y < 1, 0 ≤ t≤ 0.025

u (x,0,t) = 0, 0 < x < 1, 0 ≤ t≤ 0.025

u (1,y,t) = e-3ty, 0 < y < 1, 0 ≤ t≤ 0.025

u (x,1,t) = e-3tx2, 0 < x < 1, 0 ≤ t≤ 0.025

This fractional partial differential equation together with the above initial and boundary condition is constructed such that the exact solution is u(x,y,t) = e-3tx2y.

Table 1 and 2 show the numerical solution using the explicit finite difference approximation. From table 1 and 2, it can be seen that that good agreement between the numerical solution and exact solution.

Tables 3 show Maximum error between the exact analytical solution and the numerical solution obtained by applying the explicit Euler method discussed in this paper.

Table 1

The numerical solution of example by using the finite difference method.

$(\Delta x=0.2,\Delta y=0.2,\Delta t=0.0125)$

x = y	t	Numerical Solution	Exact Solution	|uex -uapprox.|
0.2	0.0125	4.509E-3	7.70556 E -3	3.19656 E -3
0.4	0.0125	0.055	6.16444 E -2	6.64444 E -3
0.6	0.0125	0.198	0.20805	1.00500 E -2
0.8	0.0125	0.481	0.49316	1.21555 E -2
0.2	0.0250	0.012	7.42195 E -3	4.57805 E -2
0.4	0.0250	0.105	5.93756 E -2	4.56244 E -2
0.6	0.0250	0.141	0.20039	5.93926 E -2
0.8	0.0250	0.475	0.47500	4.66500 E -6

Table 2

The numerical solution of example by using the finite difference method.

$(\Delta x=0.25,\Delta y=0.25,\Delta t=0.0125)$

x = y	t	Numerical Solution	Exact Solution	|uex -uapprox.|
0.25	0.0125	0.011	1.50499 E -2	4.04991 E -3
0.50	0.0125	0.112	0.12040	8.39930 E -3
0.75	0.0125	0.394	0.40635	1.23476 E -2
0.25	0.0250	0.020	1.44960 E -2	5.50400 E -3
0.50	0.0250	0.166	0.11597	5.00321 E -2
0.75	0.0250	0.388	0.39139	3.39178 E -3

Table 3

Maximum error for the numerical solution of example by using the finite difference method.

$\Delta x = \Delta y$	Δt	Maximum Error
0.20	0.0125	0.0121555
0.25	0.0125	0.0123476

6. Discussion

In this paper, a numerical method for solving two-dimensional two-sided fractional diffusion equation has been described and demonstrated. The explicit Euler method is proved to be conditionally stable and converges. Furthermore numerical example is presented to show that good agreement between the numerical solution and exact solution has been noted

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