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Original article

On solution of irregular differential equation with boundary conditions

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A R T I C L E I N F O

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A B S T R A C T

This paper is devoted to the analysis of irregular singular boundary value problems for ordinary differential equations with a singularity of the different kinds. We propose a semi - analytic technique using two point osculatory interpolations to construct polynomial solution, and then discuss the behavior of the solution in the neighborhood of the irregular singular points and its numerical approximation. Also we introduce an example to demonstrate the applicability and efficiency of the method.

1. Introduction

In the study of nonlinear phenomena in physics, engineering and other sciences, many mathematical models lead to singular two-point boundary value problems (SBVPs) associated with nonlinear second order ordinary differential equations (ODEs).

In mathematics, a singularity is in general a point at which a given mathematical object is not defined, or a point of an exceptional set where it fails to be well-behaved in some particular way, such as many problems in varied fields as thermodynamics, electrostatics, physics, and statistics give rise to ordinary differential equations of the form :

$$y'' = f(x, y, y'), \ a < x < b, \qquad (1)$$

on some interval of the real line with some boundary conditions.

A two-point BVP associated to the second order differential equation (1) is singular if one of the following situations occurs:

a and/or b are infinite; f is unbounded at some $x0 \in [0,1]$ or f is unbounded at some particular value of y or y′ (Robert et al., 1996) .

How to solve a linear ODE of the form:

$$A(x)y'' + B(x)y' + C(x)y = 0 \quad , \qquad (2)$$

The first thing we do is, rewrite the ODE as:

$$y'' + P(x)y' + Q(x)y = 0 \quad , \qquad (3)$$

Where, of course, $P(x) = B(x) / A(x)$, and $Q(x) = C(x) / A(x)$.

There are two types of point $x_0 \in [0,1]$: Ordinary Point and Singular Point. Also, there are two types of Singular Point: Regular and Irregular Points. A function y(x) is analytic at x_0 if it has a power series expansion at x_0 that converges to y(x) on an open interval containing x_0. A point x_0 is an ordinary point of the ODE (3), if the functions P(x) and Q(x) are analytic at x_0. Otherwise x_0 is a singular point of the ODE,

i.e. $$P(x) = P_0 + P_1(x-x_0) + P_2(x-x_0)^2 + ... = \sum_{i=0}^{\infty} p_i (x - x_0)^i \quad , \qquad (4)$$

$$Q(x) = Q_0 + Q_1(x-x_0) + Q_2(x-x_0)^2 + ... = \sum_{i=0}^{\infty} q_i (x - x_0)^i \quad , \qquad (5)$$

A singular point x0 of the ODE (3) is a regular singular point of the ODE if the functions xP(x) and $x^2Q(x)$ are analytic at x_0. Otherwise x_0 is an irregular singular point of the ODE (Rachůnková et al., 2008)

Shampine in (Shampine et al., 2000) gave other definition, which illustrated by the following:

If $\lim_{x\to 0} (x - x_0)P(x)$ finite and $\lim_{x\to 0} (x - x_0)^2 Q(x)$ finite , (6)

Now, we state the following theorem without proof which gives us a useful way of testing if a singular point is irregular.

Theorem 1 (Howell, 2009)

If the $\lim_{x\to 0} P(x)$ and $\lim_{x\to 0} Q(x)$ are exist, finite, and not equal to zero then x = 0 is a regular singular point. If both limits are zero, then x = 0 may be a regular singular point or an ordinary point. If either limit fails to exists or is $\pm\infty$, then x = 0 is an irregular singular point.

There are four kinds of singularities (Howell, 2009):

The first kind is the singularity at one of the ends of the interval [0,1] ;

The second kind is the singularity at both ends of the interval [0,1] ;

The third kind is the case of a singularity in the interior of the interval;

The forth and final kind is simply treating the case of a regular differential equation on an infinite interval.

In this paper, we focus of the first three kinds.

2. Solution of second order SBVP

In this section we suggest semi analytic technique to solve second order SBVP as following, we consider the SBVP:

$$x^m y'' + f(x, y, y') = 0 \quad , \qquad (7a)$$

$$g_i(y(0), y(1), y'(0), y'(1)) = 0, \; i = 1, 2, \qquad (7b)$$

Where f is linear function and g_1, g_2 are in general nonlinear functions of their arguments.

The simple idea behind the use of two-point polynomials is to replace y(x) in problem (7), or an alternative formulation of it, by a two point osculatory interpolation polynomial P_{2n+1} which enables any unknown boundary values or derivatives of y(x) to be computed .

The first step therefore is to construct the P_{2n+1}, to do this we need the Taylor coefficients of y(x) at x = 0:

$$y = a_0 + a_1 x + \sum_{i=2}^{\infty} a_i x^i \quad , \qquad (8)$$

Where $y(0)= a_0$,$y'(0)= a_1$,$y''(0) / 2! =a_2$,..., $y^{(i)}(0) / i! = a_i$, i= 3, 4,...

Then insert the series forms (8) into (7a) and equate coefficients of powers of x to obtain a2. Also we need Taylor coefficient of y(x) about x = 1:

$$y = b_0 + b_1 (x-1) + \sum_{i=2}^{\infty} b_i (x-1)^i \quad , \quad (9)$$

Where $y(1) = b_0$, $y'(1) =b_1$, $y''(1) / 2! =b_2$, ... , $y^{(i)}(1) / i! =b_i$, i = 3, 4,...

then insert the series form (9) into (7a) and equate coefficients of powers of (x-1) to obtain b_2 ,then derive equation (7a) with respect to x and iterate the above process to obtain a_3 and b_3 , now iterate the above process many times to obtain a_4 , b_4 ,then a_5 , b_5 and so on, that is ,we can get a_i and b_i for all i ≥ 2(the resulting equations can be solved using MATLAB to obtain ai and bi for all i ≥ 2) , the notation implies that the coefficients depend only on the indicated unknowns a_0, a_1, b_0, b_1, we get two of these four unknown by the boundary condition . Now, we can construct a two point osculatory interpolation polynomial P_{2n+1} from these coefficients (a_i's and b_i's) by the following: (Morgado, et al., 2009)

$$P_{2n+1} = \sum_{i=0}^{n} \{ a_i Q_i(x) + (-1)^i b_i Q_i(1-x)\} \quad , \quad (10)$$

Where $\quad Q_j(x)/ j! = (x^j/j!)(1-x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} x^s$

We see that (10) have only two unknowns from a_0 ,b_0 ,a_1 and b_1 to find this, we integrate equation (7a) on [0, x] to obtain :

$$x^m y'(x) - m x^{m-1}y(x) + m(m-1)\int_0^x x^{m-2}y(x)\, dx + \int_0^x f(x, y, y')\, dx = 0 \quad , \quad (11a)$$

And again integrate equation (11a) on [0, x] to obtain:

$$x^m y(x) - 2m\int_0^x x^{m-1}y(x)dx + m(m-1)\int_0^x (1-x)x^{m-2}\, y(x)\, dx + \int_0^x (1-x)f(x,y,y') = 0, \quad (11b)$$

Putting x = 1 in (11) then gives that:

$$b_1 - mb_0 + m(m-1)\int_0^1 x^{m-2}\, y(x)\, dx + \int_0^1 f(x, y, y')\, dx = 0 \quad , \qquad (12a)$$

And

$$b_0 - 2m\int_0^1 x^{m-1}y(x)dx + m(m-1)\int_0^1 (1-x)x^{m-2}\, y(x)dx + \int_0^1 (1-x)f(x, y, y')dx = 0 \; , \quad (12b)$$

Use P_{2n+1} as a replacement of y(x) in (12) and substitute the boundary conditions(7b) in(12), then we have only two unknown coefficients b_1, b_0 and two equations(12). So, we can find b_1, b_0 for any n by solving this system of algebraic Equations using MATLAB, so insert b_0 and b_1 into (10), which represents the solution of (7).

Practical computations have shown that this generally provides a more accurate polynomial representation for a given n.

Now we introduce an example of second order SBVP, non homogenous, linear ODE with irregular singular point which illustrates suggested method.

Example

Consider the following SBVP:

$$x^2 y'' + (1+3x)\, y' + y = 0 \quad , \quad 0 \le x \le 1$$

with BC: $y'(0) = -y(0)$, $y'(1) = 1$

It is clear that x = 0, is irregular singular point of the first kind.

Now, we solve this example using semi-analytic technique as shown in the following:

From equations (10) we have:

$$P_5 = 4.3076923077\, x^5 - 14.1538461538\, x^4 + 17.3846153846\, x^3 -10.7179487180\, x^2 + 5.3589743590\, x - 5.3589743590.$$

Higher accuracy can be obtained by evaluating higher n, now, we take n= 3, i.e.,

$$P_7 = 12.0670391061\, x^7 - 50.7581803668\, x^6 + 83.5434956105\, x^5 - 68.1947326418\, x^4 + 30.4565043895\, x^3 - 10.1521681298\, x^2 + 5.0760840649\, x - 5.0760840649$$

Now, increase n, to get higher accuracy, let n = 4 , i.e. ,

$$P_9 = - 93.3587300498\, x^9 +362.8396681281\, x^8 - 487.8590810224\, x^7 + 222.5363114349\, x^6 + 18.3004913637x^5-9.2782131392x^3+ 9.7594043797x^2 - 4.8797021898\, x + 4.879702189848218$$

For more details, table (1) give the results for different nodes in the domain, for n = 2, 3, 4, i.e. P^5, P^7, P^9 and figure (1) illustrate suggested method for n= 4, i.e., P_9.

Table 1

The result of the method for n= 2, 3, 4 of example.

	P_5	P_7	P_9
a0	-5.358974358974359	-5.076084064910881	-4.879702189848218
b0	-3.179487179487179	-3.03804203245544	-2.939851094924109
xi	P_5	P_7	P_9
0	-5.358974358974359	-5.076084064910881	-4.879702189848218
0.1	-4.914244102564102	-4.645574425112895	-4.460408091878864
0.2	-4.598088205128205	-4.308773661079556	-4.074647176770410
0.3	-4.350691282051282	-4.028358650704274	-3.625598887089984
0.4	-4.135868717948719	-3.799215689278588	-3.128434992594118
0.5	-3.935897435897436	-3.619280393721494	-2.692547882262536
0.6	-3.746346666666668	-3.478678133544401	-2.443413018665542
0.7	-3.570908717948719	-3.361083200842198	-2.432859981745726
0.8	-3.416229743589745	-3.251214931600992	-2.597311419442051
0.9	-3.286740512820513	-3.142388990612921	-2.800456096971161
1	-3.179487179487179	-3.038042032288633	-2.939851095007586

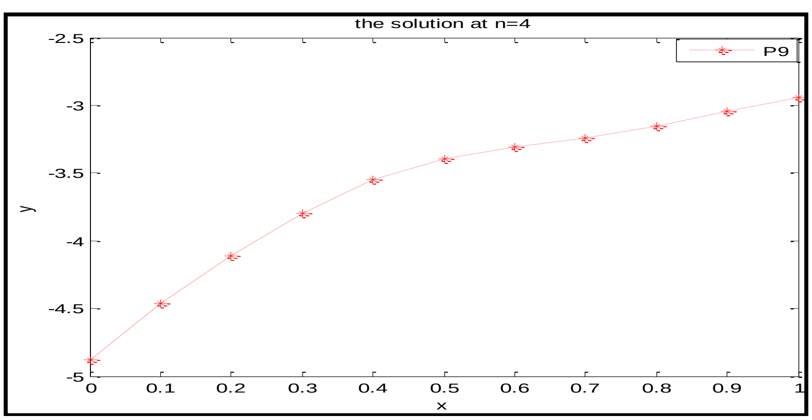

Fig. 1. Illustrate suggested method for n = 4, i.e., P_9.

3. Behavior of the solution in the neighborhood of the singularity x= 0

Our main concern in this section will be the study of the behavior of the solution in the neighborhood of singular point.

Consider the following SIVP:

$y''(x) + ((N - 1) / x)\, y'(x) = f(y),\ N \geq 1, \qquad 0 < x < 1, \qquad (13)$

$y(0) = y_0\ ,\quad \lim_{x\to 0^+} x\, y'(x) = 0\ , \qquad (14)$

where f(y) is continuous function .

As the same manner in (Burden et al., 2001), let us look for a solution of this problem in the form:

$y(x) = y_0 - C x^k (1 + o(1))\ , \qquad (15)$

$y'(x) = - C k x^k-1(1 + o(1))\ ,$

$y''(x) = - C k (k - 1) x^{k-2}(1 + o(1))\ , \qquad x \to 0^+$

where C is a positive constant and k > 1. If we substitute (15) in (13) we obtain:

$C = (1/ k) (f (y0) / N) k^{-1}\ , \qquad (16)$

In order to improve representation (15) we perform the variable substitution:

$y (x) = y_0 - C x^k (1 + g(x))\ , \qquad (17)$

we easily obtain the following result which is similar to the results in (Burden et al., 2001).

Theorem 2 (Rasheed, 2011)

For each $y_0 > 0$, problem (13), (14) has, in the neighborhood of x = 0, a unique solution that can be represented by:

$y(x, y_0) = y_0 - C x^k (1 + g x^k + o(x^k))\ ,$

where k, C and g are given by (16) and (17), respectively.

We see that these results are in good agreement with the ones obtained by the method in (Burden et al., 2001), they are also consistent with the results presented in (Morgado, et al., 2009).

In order to estimate the convergence order of the suggested method at x = 0, we have carried out several experiments with different values of n and used the formula

$cy_0 = -\log_2(|y_0^{n3} - y_0^{n2}| / |y_0^{n2} - y_0^{n1}|)\ , \qquad (18)$

where y_0^{ni} is the approximate value of y_0 obtained with n_i ,n_i = 1,2, 3, 4,…

Now, we apply above study to the previous example.

Let y_{0i} is the approximate value of y0 evaluated by suggested method with n = i, i = 2, 3, 4, by:

P_{2n+1}	y_{0i}
P5	-5.358974358974359
P7	-5.076084064910881
P9	-4.879702189848218

$$C_{y0} = -\log_2 \frac{|y_{04} - y_{03}|}{|y_{03} - y_{02}|}$$

$C_{y0} = -\ \log_2 (0.196381875062663 / 0.282890294063478)$;

$C_{y0} = 0.526580895669647$

The result of C_{y0} illustrate that the convergence order at x = 0 estimate of this example is close to one.

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