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On the triangle inequality in quasi 2-normed spaces and a new characterization of 2-inner product spaces

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A B S T R A C T

In this paper we establish a generalization of the triangle inequality in quasi 2-normed linear spaces. Moreover, we obtained a new characterization of 2-inner product spaces by comparing the 2-angular distance and 2-skew-angular distance with each other.

1. Introduction and preliminaries

The triangle inequality is one of the most fundamental inequality in calculus. Many interesting refinements and reverses of this inequality in normed linear spaces have been obtained (see for instance Dragomir, 2009; Kato et al. 2007; Maligranda, 2008; Najati and Mohammadi Saem, 2013; Pečaric and Rajic′ 2013).

Theory of linear 2-normed spaces has been introduced by Gähler 1965, and has been developed extensively in different subjects by others(see Freese and Cho, 2001,).

Definition 1.1 Let X be a linear space of dimension greater than 1 over filed $\mathbb{R}$ of real numbers. Suppose $\|.,.\|$is a real-valued function on $X \times X$ satisfying the following conditions:

1) $\| x, y \| = 0$ if and only if x and y are linearly dependent vectors.
2) $\| x, y \| = \| y, x \|$ for all $x, y \in X$.
3) $\| \lambda x, y \| = |\lambda| \, \| x, y \|$ for all $\lambda \in \mathbb{R}$ and $x, y \in X$.

4) $\| x+y,z \| \leq \| x,z \| + \| y,z \|$ for all $x,y,z \in X$.

Then $\|.,.\|$ is called a 2-norm on X and $(X,\|.,.\|)$ is called a linear 2-normed space.

It is known that equality holds for every $z \in X$ in definition (1.1) part (4) if and only if x and y are linearly dependent with the same direction. It is easy to show that the 2-norm $\|.,.\|$ is non-negative and $\| x,y+\alpha x \| = \| x,y \|$ for all $x,y \in X$ and $\alpha \in \mathbb{R}$.

A concept which is closely related to 2-normed linear space is 2-inner product space that introduced by Diminnie, Gähler and White 1973, 1977 .

Definition 1.2 Let X be a linear space of dimension greater than 1 over filed $\mathbb{R}$. Suppose that $\langle .,.|.\rangle$ is a $\mathbb{R}$-valued function defined on $X \times X \times X$ satisfying the following conditions:

1)$\langle x,x|z\rangle \geq 0$ and $\langle x,x|z\rangle = 0$ if and only if x and z are linearly dependent.

2)$\langle x,x|z\rangle = \langle z,z|x\rangle$

3)$\langle y,x|z\rangle = \langle x,y|z\rangle$

4)$\langle \alpha x,y|z\rangle = \alpha\langle x,y|z\rangle$ for any scalar $\alpha \in \mathbb{R}$

5)$\langle x+x',y|z\rangle = \langle x,y|z\rangle + \langle x',y|z\rangle$ $\langle .,.|.\rangle$ is called a 2-inner product and $(X,\langle .,.|.\rangle)$ is called a 2-inner product space(or a 2-per-Hilbert space). Also, $\langle .,.|.\rangle$ is 2-inner product on X and $(X,\langle .,.|.\rangle)$ is called a 2-inner product space.

We refer to Cho et al. 2001, for some basic properties of 2-inner products spaces. Choonkil Park 2006, introduced the notion of quasi-2-normed spaces as follows:

Definition 1.3 Let X be a linear space. A quasi-2-norm is a real-valued function on $X \times X$ satisfying the following properties:

1) $\| x,y \| = 0$ if and only if x and y are linearly dependent vectors.

2) $\| x,y \| = \| y,x \|$ for all $x,y \in X$.

3) $\| \lambda x,y \| = |\lambda| \, \| x,y \|$ for all $\lambda \in \mathbb{R}$ and $x,y \in X$.

4)There is a constant $K \geq 1$ such that $\| x+y,z \| \leq K \| x,z \| + K \| y,z \|$ for all $x,y,z \in X$.

The pair $(X,\|.,.\|)$ is called a quasi-2-normed space if $\|.,.\|$ is a quasi-2-norm on X. The smallest possible K is called the modulus of concavity of $\|.,.\|$.

It follows from (4) that

$$\frac{1}{K} \| x,z \| - \| y,z \| \leq \| x-y,z \|,$$

and

$$\frac{1}{K} \| y,z \| - \| x,z \| \leq \| x-y,z \|,$$

for all $x,y,z \in X$.

The aim of this paper is to present some refitment of the triangle inequality in quasi 2-normed spaces and obtain a new characterization of 2-inner product spaces.

2. Some refinement of the triangle inequality in quasi 2-normed spaces

In this section, we establish some generalization of the triangle inequality and its reverse in quasi 2-normed spaces.

Theorem 2.1 For non-zero vectors x,y and z in a quasi-2-normed space X with $z \notin \operatorname{span}\{x,y\}$, we have

$$\| x+y,z \| \leq K(\| x,z \| + \| y,z \|) - K\left(2 - \left\| \frac{x}{\|x,z\|} + \frac{y}{\|y,z\|}, z \right\|\right)\min\{\| x,z \|, \| x,z \|\} \qquad (1)$$

and

$$\| x+y,z \| \geq (\| x,z \| + \| y,z \|) - \left(2 - \frac{1}{K}\left\| \frac{x}{\|x,z\|} + \frac{y}{\|y,z\|}, z \right\|\right)\max\{\| x,z \|, \| y,z \|\}. \qquad (2)$$

where $K \geq 1$.

Proof. Without loss of generality we may assume that $\| x,z \| \leq \| y,z \|$. Then we have

$$\| x+y,z \| = \left\|\| x,z \| \left(\frac{x}{\|x,z\|} + \frac{y}{\|y,z\|}\right) + \| y,z \| \frac{y}{\|y,z\|} - \| x,z \| \frac{y}{\|y,z\|}, z \right\|$$

$$\leq K \left\|\| x,z \| \left(\frac{x}{\|x,z\|} + \frac{y}{\|y,z\|}\right), z \right\| + K \left\|\| y,z \| \frac{y}{\|y,z\|} - \| x,z \| \frac{y}{\|y,z\|}, z \right\|$$

$$= K \| x,z \| \left\| \frac{x}{\|x,z\|} + \frac{y}{\|y,z\|}, z \right\| + K(\| y,z \| - \| x,z \|)$$

$$= K \| x,z \| \left\| \frac{x}{\|x,z\|} + \frac{y}{\|y,z\|}, z \right\| + K(\| y,z \| + \| x,z \| - 2 \| x,z \|)$$

$$= K \| x,z \| \left(\left\| \frac{x}{\|x,z\|} + \frac{y}{\|y,z\|}, z \right\| - 2\right) + K(\| y,z \| + \| x,z \|).$$

which establishes the estimate (1). Now since

$$\begin{aligned}
\| x+y,z \| &= \| \| y,z \| (\tfrac{x}{\|x,z\|}+\tfrac{y}{\|y,z\|}) - (\| y,z \| \tfrac{x}{\|x,z\|} - \| x,z \| \tfrac{x}{\|x,z\|}),z \| \\
&\geq \tfrac{1}{K} \| \| y,z \| (\tfrac{x}{\|x,z\|}+\tfrac{y}{\|y,z\|}),z \| - \| \| y,z \| \tfrac{x}{\|x,z\|} - \| x,z \| \tfrac{x}{\|x,z\|},z \| \\
&= \tfrac{1}{K} \| y,z \| \| \tfrac{x}{\|x,z\|}+\tfrac{y}{\|y,z\|},z \| - (\| y,z \| - \| x,z \|) \\
&= \tfrac{1}{K} \| y,z \| \| \tfrac{x}{\|x,z\|}+\tfrac{y}{\|y,z\|},z \| + (\| y,z \| + \| x,z \| - 2\| y,z \|) \\
&= \| y,z \| (\tfrac{1}{K} \| (\tfrac{x}{\|x,z\|}+\tfrac{y}{\|y,z\|}),z \| - 2) + (\| y,z \| + \| x,z \|).
\end{aligned}$$

gives the inequality (2).

Following we give a refinement of inequality (2).

Corollary 2.2 For non-zero vectors x, y and z in a quasi-2-normed space X with $z \notin \mathrm{span}\{x,y\}$, we have

$$\| x+y,z \| \geq K(\| x,z \| + \| y,z \|) - (2K^2 - \| \tfrac{x}{\|x,z\|}+\tfrac{y}{\|y,z\|},z \|)\max\{\| x,z \|, \| y,z \|\}. \qquad (3)$$

where $K \geq 1$.

Proof. Without loss of generality we may assume that $\| x,z \| \leq \| y,z \|$. Then we have

$$\begin{aligned}
K(\| x,z \| + \| y,z \|) &\leq K \| x+y,z \| + (2K - \| \tfrac{x}{\|x,z\|}+\tfrac{y}{\|y,z\|},z \|) \| y,z \| \\
&= \| x+y,z \| + (K-1) \| x+y,z \| + (2K - \| \tfrac{x}{\|x,z\|}+\tfrac{y}{\|y,z\|},z \|) \| y,z \| \\
&\leq \| x+y,z \| + (K-1)K(\| x,z \| + \| y,z \|) + (2K - \| \tfrac{x}{\|x,z\|}+\tfrac{y}{\|y,z\|},z \|) \| y,z \| \\
&\leq \| x+y,z \| + (K-1)K(2\| y,z \|) + (2K - \| \tfrac{x}{\|x,z\|}+\tfrac{y}{\|y,z\|},z \|) \| y,z \| \\
&= \| x+y,z \| + (2K^2 - \| \tfrac{x}{\|x,z\|}+\tfrac{y}{\|y,z\|},z \|) \| y,z \|.
\end{aligned}$$

gives the inequality (3).

Now, we can used (1) and (2) for the following estimation of the 2-angular distance

$$\alpha[x,y;z] = \| \tfrac{x}{\|x,z\|} - \tfrac{y}{\|y,z\|},z \|$$

between non-zero elements x, y and z in X with $z \notin \mathrm{span}\{x,y\}$.

Corollary 2.3 For non-zero vectors x, y and z in a quasi-2-normed space X with $z \notin \mathrm{span}\{x,y\}$, we have

$$\frac{\|x-y,z\| - K|\|y,z\| - \|x,z\||}{K\min\{\|x,z\|,\|y,z\|\}} \leq \alpha[x,y;z] \leq \frac{K\|x-y,z\| + K|\|y,z\| - \|x,z\||}{\max\{\|x,z\|,\|y,z\|\}}.$$

where $K \geq 1$.

Theorem 2.4 For non-zero vectors x, y and z in a quasi-2-normed space X with $z \notin \mathrm{span}\{x,y\}$, we have

$$\| x+y,z \| \leq K(\| x,z \| + \| y,z \|) - K(\tfrac{\|x,z\|}{\|y,z\|} + \tfrac{\|y,z\|}{\|x,z\|} - \| \tfrac{x}{\|y,z\|}+\tfrac{y}{\|x,z\|},z \|)\min\{\| x,z \|, \| x,z \|\} \qquad (4)$$

and

$$\| x+y,z \| \geq \| x,z \| + \| y,z \| - (\tfrac{\|x,z\|}{\|y,z\|} - \tfrac{\|y,z\|}{\|x,z\|} - \tfrac{1}{K} \| \tfrac{x}{\|y,z\|}+\tfrac{y}{\|x,z\|},z \|)\max\{\| x,z \|, \| y,z \|\}. \quad (5)$$

where $K \geq 1$.

Proof. Without loss of generality we may assume that $\| x,z \| \leq \| y,z \|$. Then we have

$$\begin{aligned}
\| x+y,z \| &= \| \tfrac{\|x,z\|}{\|y,z\|}x + \tfrac{\|x,z\|}{\|x,z\|}y + (1 - \tfrac{\|x,z\|}{\|y,z\|})x,z \| \\
&\leq K \| x,z \| \| \tfrac{x}{\|y,z\|}+\tfrac{y}{\|x,z\|},z \| + K \| x,z \| - K \tfrac{\|x,z\|^2}{\|y,z\|} \\
&= K \| x,z \| + K \| x,z \| (\| \tfrac{x}{\|y,z\|}+\tfrac{y}{\|x,z\|},z \| - \tfrac{\|x,z\|}{\|y,z\|}) \\
&= K \| x,z \| + K \| y,z \| + K \| x,z \| (\| \tfrac{x}{\|y,z\|}+\tfrac{y}{\|x,z\|},z \| - \tfrac{\|x,z\|}{\|y,z\|} - \tfrac{\|y,z\|}{\|x,z\|})
\end{aligned}$$

which establishes the estimate (3). Similarly, we have

$$\begin{aligned}
\| x+y,z \| &= \| \tfrac{\|y,z\|}{\|y,z\|}x + \tfrac{\|y,z\|}{\|x,z\|}y + (1 - \tfrac{\|y,z\|}{\|x,z\|})y,z \| \\
&\geq \tfrac{1}{K} \| y,z \| \| \tfrac{x}{\|y,z\|}+\tfrac{y}{\|x,z\|},z \| - (\tfrac{\|y,z\|^2}{\|x,z\|} - \| y,z \|) \\
&= \| y,z \| + \| y,z \| (\tfrac{1}{K} \| \tfrac{x}{\|y,z\|}+\tfrac{y}{\|x,z\|},z \| - \tfrac{\|y,z\|}{\|x,z\|}) \\
&= \| x,z \| + \| y,z \| + \| y,z \| (\tfrac{1}{K} \| \tfrac{x}{\|y,z\|}+\tfrac{y}{\|x,z\|},z \| - \tfrac{\|x,z\|}{\|y,z\|} - \tfrac{\|y,z\|}{\|x,z\|}).
\end{aligned}$$

gives inequality (4).

Now, we can used (4) and (5) for the following estimation of the 2-skew-angular distance

$$\beta[x,y;z] = \| \frac{x}{\|y,z\|} - \frac{y}{\|x,z\|}, z \|$$

between non-zero elements x, y and z in X with $z \notin \operatorname{span}\{x,y\}$.

Corollary 2.5 For non-zero vectors x, y and z in a quasi-2-normed space X with $z \notin \operatorname{span}\{x,y\}$, we have

$$\beta[x,y;z] \leq K \frac{\|x-y,z\|}{\max\{\|x,z\|,\|y,z\|\}} + K \frac{|\|y,z\|-\|x,z\||}{\min\{\|x,z\|,\|y,z\|\}}, \qquad (6)$$

and

$$\beta[x,y;z] \geq \frac{1}{K} \frac{\|x-y,z\|}{\min\{\|x,z\|,\|y,z\|\}} - \frac{|\|y,z\|-\|x,z\||}{\max\{\|x,z\|,\|y,z\|\}}. \quad (7)$$

where $K \geq 1$.

Proof. Without loss of generality we may assume that $\| x,z \| \leq \| y,z \|$. Estimate (5) implies that

$$\| \frac{x}{\|y,z\|} - \frac{y}{\|x,z\|}, z \| \max\{\| x,z \|, \| y,z \|\}$$
$$\leq K \| x-y,z \| - K \| x,z \| - K \| y,z \| + K\left(\frac{\|x,z\|}{\|y,z\|} - \frac{\|y,z\|}{\|x,z\|}\right)\max\{\| x,z \|, \| y,z \|\}.$$

Then

$$\| \frac{x}{\|y,z\|} - \frac{y}{\|x,z\|}, z \| \| y,z \| \leq K \| x-y,z \| + K\frac{\|y,z\|}{\|x,z\|}(\| y,z \| - \| x,z \|),$$

and so

$$\| \frac{x}{\|y,z\|} - \frac{y}{\|x,z\|}, z \| \leq K\frac{\|x-y,z\|}{\|y,z\|} + K\frac{|\|y,z\|-\|x,z\||}{\|x,z\|}.$$

Similarly, inequality (4) implies that

$$\| \frac{x}{\|y,z\|} - \frac{y}{\|x,z\|}, z \| \| x,z \| \geq \frac{1}{K} \| x-y,z \| - \frac{\|x,z\|}{\|y,z\|}(\| y,z \| - \| x,z \|),$$

and so

$$\| \frac{x}{\|y,z\|} - \frac{y}{\|x,z\|}, z \| \geq \frac{1}{K}\frac{\|x-y,z\|}{\|x,z\|} - \frac{|\|y,z\|-\|x,z\||}{\|y,z\|},$$

which completes the proof.

3. A characterization of 2-inner product spaces

In this section we compare that the $\alpha[x,y;z]$ and $\beta[x,y;z]$ and will present a necessary and sufficient condition for a 2-normed space to be a 2-inner product space.

To prove that, we need the following theorem by Soenjaya 2012.

Theorem 3.1 Let $x, y, z \in X$ and $\alpha \in \mathbb{R}$. The following statements are equivalent:

(i) X is a 2-inner product space.

(ii) $\| x,z \| = \| y,z \|$ implies $\| x+y,z \| \leq \| \alpha x, \alpha^{-1}y, z \|$ for all $\alpha \neq 0$.

(iii) $\| x+y,z \| \leq \| \alpha x + \alpha^{-1}y, z \|$ for all $\alpha \neq 0$, implies $\| x,z \| = \| y,z \|$.

Theorem 3.2 Let X be a real 2-normed linear space. Then X is a 2-inner product space if and only if for each linear independent vectors x, y and z in X,

$$\alpha[x,y;z] \leq \beta[x,y;z]. \qquad (8)$$

Proof. Let X be an 2-inner product space, x, y and z are non-zero vectors in X with $z \notin \operatorname{span}\{x,y\}$. Then we have

$$\| \frac{x}{\|y,z\|} - \frac{y}{\|x,z\|}, z \|^2 - \| \frac{x}{\|x,z\|} - \frac{y}{\|y,z\|}, z \|^2$$
$$= \langle \frac{x}{\|y,z\|} - \frac{y}{\|x,z\|}, \frac{x}{\|y,z\|} - \frac{y}{\|x,z\|} | z \rangle - \langle \frac{x}{\|x,z\|} - \frac{y}{\|y,z\|}, \frac{x}{\|x,z\|} - \frac{y}{\|y,z\|} | z \rangle$$
$$= \frac{\|x,z\|^2}{\|y,z\|^2} + \frac{\|y,z\|^2}{\|x,z\|^2} - \frac{2\langle x,y|z\rangle}{\|x,z\|\|y,z\|} - \left(\frac{\|x,z\|^2}{\|x,z\|^2} + \frac{\|y,z\|^2}{\|y,z\|^2} - \frac{2\langle x,y|z\rangle}{\|x,z\|\|y,z\|}\right)$$
$$= \frac{\|x,z\|^2}{\|y,z\|^2} + \frac{\|y,z\|^2}{\|x,z\|^2} - 2$$
$$= \left(\frac{\|x,z\|}{\|x,z\|} - \frac{\|y,z\|}{\|y,z\|}\right)^2 \geq 0,$$

which prove the necessity.

To prove the sufficiency, let $x, y, z \in X$, $\| x, z \| = \| y, z \|$ and $\alpha \neq 0$. From theorem (3.1) it is enough to prove that

$$\| x + y, z \| \leq \| \alpha x + \alpha^{-1} y, z \|.$$

If x, z or y, z be linearly dependent, then proof is clear. Let x, z and y, z are mutually linear independent and $\alpha > 0$. Applying inequality (8) to $\alpha^{1/2} x$ and $-\alpha^{-1/2} y$ instead of x and y, respectively, we obtain

$$\| \frac{\alpha^{1/2} x}{\alpha^{1/2} \| x, z \|} + \frac{\alpha^{-1/2} y}{\alpha^{-1/2} \| y, z \|}, z \| \leq \| \frac{\alpha^{1/2} x}{\alpha^{-1/2} \| y, z \|} + \frac{\alpha^{1/2} y}{\alpha^{-1/2} \| y, z \|}, z \|.$$

Thus

$$\| \frac{x}{\| x, z \|} + \frac{y}{\| y, z \|}, z \| \leq \| \frac{\alpha x}{\| y, z \|} + \frac{\alpha^{-1} y}{\| y, z \|}, z \|.$$

Since $\| x, z \| = \| y, z \| \neq 0$, we have

$$\| x + y, z \| \leq \| \alpha x + \alpha^{-1} y, z \|.$$

Now, let α be negative. Put $\beta = -\alpha > 0$. From the positive case, we obtain

$$\| x + y, z \| \leq \| \beta x + \beta^{-1} y, z \| = \| \alpha x + \alpha^{-1} y, z \|,$$

which complete the proof.

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