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Original article

ON functors $\mathbf{D_d(-)}$ and $\mathbf{D_d(M,-)}$

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Let R be a Noetherian ring and d be a non-negative integer. Let M be an R-module. We define the d- transform functors $\mathrm{D_d}(-)$ and $\mathrm{D_d}(\mathrm{M},-)$ on the category of R-modules. We show that, if M is an injective R- module, then $\mathrm{L_d}(\mathrm{M})$ is injective R- module. Also, a criterion under which the isomorphism $\mathrm{M} \cong \mathrm{D_d}(\mathrm{M})$ holds will be investigated. Finally, we investigate $\mathrm{D_d}(\mathrm{M},\mathrm{N})$, where M and N are R-modules.

1. Introduction

Throughout this paper, R denotes a commutative Noetherian ring with unity and d will be a non-negative integer. For R-modules M and N, we define

1) $$\mathrm{L_d}(\mathrm{M}) = \{\mathrm{m} \in \mathrm{M} : \dim \mathrm{Rm} \leq \mathrm{d}\}$$

2) $$\mathrm{H}^{\mathrm{i}}_{\mathrm{d}}(\mathrm{M}) = \varinjlim_{\dim \frac{\mathrm{R}}{\mathfrak{a}} \leq \mathrm{d}} \mathrm{Ext}^{\mathrm{i}}_{\mathrm{R}}\left(\frac{\mathrm{R}}{\mathfrak{a}}, \mathrm{M}\right) \quad \text{for all } \mathrm{i} \geq 0$$

3) $$L_d(M,N) = \varinjlim_{\dim\frac{R}{\mathfrak{a}} \le d} \mathrm{Hom}_R\left(\frac{R}{\mathfrak{a}M}, N\right)$$

4) $$H_d^i(M,N) = \varinjlim_{\dim\frac{R}{\mathfrak{a}} \le d} \mathrm{Ext}_R^i\left(\frac{M}{\mathfrak{a}M}, N\right) \quad \text{for all } i \ge 0$$

It can be shown easily that $L_d(-)$ and $L_d(M,-)$ are additive covariant R- linear functors on the category of R- modules which are left exact, too. For R- modules M and N, we easily have $H_d^i(M) \cong \mathcal{R}^i(L_d(M))$ and $H_d^i(M,N) \cong \mathcal{R}^i(L_d(M,N))$ for all $i \ge 0$.

Using the results in [7, Theorem 2.75], it can be shown easily that

$L_d(M,N) \cong L_d(\mathrm{Hom}_R(M,N))$,

Moreover, if M be finitely generated R- module, then by [5, Satz 3]

$$L_d(M,N) \cong \mathrm{Hom}_R(M, L_d(N)).$$

2. Main results

the functors ,In the section$D_d(-)$ and $D_d(M,-)$ where ,are introduced and the realted theorems are proven M is an R- module.

Defintion 2.1. For any R- module M andN, we define

i) $$D_d(M) = \varinjlim_{\dim\frac{R}{\mathfrak{a}} \le d} \mathrm{Hom}_R(\mathfrak{a}, M)$$

ii) $$D_d(M,N) = \varinjlim_{\dim\frac{R}{\mathfrak{a}} \le d} \mathrm{Hom}_R(\mathfrak{a}M, N)$$

$D_d(-)$ and $D_d(M,-)$ are additive covariant R- linear functors which are left exact too.

Lemma 2.2. Let M be an R- module. Then the following sequence is exact:

$$0 \to L_d(M) \to M \to D_d(M) \to H_d^i(M) \to 0$$

also, for any $i \ge 1$, $R^i(D_d(M)) \cong H_d^{i+1}(M)$.

proof: Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R such that $\mathfrak{a} \subseteq \mathfrak{b}$, $\dim\frac{R}{\mathfrak{a}} \le d$ and $\dim\frac{R}{\mathfrak{b}} \le d$.

So, there is the following commutative diagram of R- modules and R- homomorphisms with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{a} \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & \frac{R}{\mathfrak{a}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathfrak{b} \arrow[r] & R \arrow[r] & \frac{R}{\mathfrak{b}} \arrow[r] & 0
\end{tikzcd}$$

Thus, we get the following commutative diagrams of R- modules and R- homomorphisms with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(\frac{R}{\mathfrak{b}}, M) \arrow[r] \arrow[d] & \mathrm{Hom}_R(R, M) \arrow[r] \arrow[d] & \mathrm{Hom}_R(\mathfrak{b}, M) \arrow[r] \arrow[d] & \mathrm{Ext}_R^1(\frac{R}{\mathfrak{b}}, M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(\frac{R}{\mathfrak{a}}, M) \arrow[r] & \mathrm{Hom}_R(R, M) \arrow[r] & \mathrm{Hom}_R(\mathfrak{a}, M) \arrow[r] & \mathrm{Ext}_R^1(\frac{R}{\mathfrak{a}}, M) \arrow[r] & 0
\end{tikzcd}$$

and for any $i \ge 1$,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ext}^i_R(\mathfrak{b}, M) & \longrightarrow & \mathrm{Ext}^{i+1}_R(\frac{R}{\mathfrak{b}}, M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ext}^i_R(\mathfrak{a}, M) & \longrightarrow & \mathrm{Ext}^{i+1}_R(\frac{R}{\mathfrak{a}}, M) & \longrightarrow & 0 \end{array}$$

Now, by applying direct limit in above commutative diagrams, the result follows.

Corollary 2.3. If R is the quotient of a catenary, biequidimensional ring, then for any $\mathfrak{p} \in \mathrm{Spec}(R)$ $D_d(M)_\mathfrak{p} \cong D_{d-\dim\frac{R}{\mathfrak{p}}}(M_\mathfrak{p})$.

Proof. Using [1, Lemma], $H^i_d(M)_\mathfrak{p} \cong H^i_{d-\dim\frac{R}{\mathfrak{p}}}(M_\mathfrak{p})$ for all $i \geq 0$. Now, by commutative diagram with exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_d(M)_\mathfrak{p} & \longrightarrow & M_\mathfrak{p} & \longrightarrow & D_d(M)_\mathfrak{p} & \longrightarrow & H^i_d(M)_\mathfrak{p} & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow id & & \downarrow & & \downarrow\cong & & \\ 0 & \longrightarrow & L_{d-\dim\frac{R}{\mathfrak{p}}}(M_\mathfrak{p}) & \longrightarrow & M_\mathfrak{p} & \longrightarrow & D_{d-\dim\frac{R}{\mathfrak{p}}}(M_\mathfrak{p}) & \longrightarrow & H^1_{d-\dim\frac{R}{\mathfrak{n}}}(M_\mathfrak{p}) & \longrightarrow & 0 \end{array}$$

and five lemma, the proof is complete.

Theorem 2.4. If E is an injective R-module, then $L_d(E)$ is also an injective R-module.

Proof. Let J be an ideal of R and let $h: J \rightarrow L_d(E)$ be an R-homomorphism. We show that $\exists\, x \in L_d(E)$ such that $h(j) = jx$ for all $j \in J$. Since E is injective, $\exists\, e \in E$ such that $h(j) = je$ for all $j \in J$. Now let $J =< j_1, \ldots, j_n >$. Thus for $1 \leq i \leq n$, there exists ideal $\mathfrak{a}_i$ Such that $\dim\frac{R}{\mathfrak{a}_i} \leq d$ and$\mathfrak{a}_i h(j_i) = 0$. Put $\mathfrak{a} = \prod_{i=1}^n \mathfrak{a}_i$. Then $\dim\frac{R}{\mathfrak{a}} \leq d$ and $\mathfrak{a}h(J) = 0$. Since $h(J) \leq Re$, using Artin-Rees lemma, $\exists c \in N$ such that for all integers $m \geq c$,

$$\mathfrak{a}^m e \cap h(J) = \mathfrak{a}^{m-c}(\mathfrak{a}^c{}^\wedge e \cap h(J)).$$

Now for $m = c+1$, we have $\mathfrak{a}^{c+1}e \cap h(J) \subseteq \mathfrak{a}h(J) = 0$.

Consequently the map $\hat{h}: \mathfrak{a}^{c+1} + J \rightarrow L_d(E)$ with $\hat{h}(r+s) = se$,

for all $r \in \mathfrak{a}^{c+1}$ and $s \in J$ is a homomorphism of R-module. Since E is injective, $\exists\, x \in E$ such that $\hat{h}(r) = rx$ for all $r \in \mathfrak{a}^{c+1} + J$. It is easy to see that for all $r \in \mathfrak{a}^{c+1}$, $x \in L_d(E)$ and the proof is complete.

Corollary 2.5. Let M , N be two R-modules and let $E^\bullet$ be an injective resolution of N. If Mis finitely generated R- module and $L_d(N) = N$, then

i) $H^i_d(N) = 0$ for all $i \geq 1$.

ii) $H^i_d(M, N) \cong \mathrm{Ext}^i_R(M, N)$ for all $i \geq 0$.

Proof. Using Theorem 2.2, we can construct an injective resolution

$$0 \rightarrow N \rightarrow E^0 \rightarrow E^1 \rightarrow \ldots$$

for N such that $L_d(E^i) = E^i$ For all $i \geq 0$. Now,

i) For each $i \geq 1$, $H^i_d(N) = \frac{\ker(E^i \rightarrow E^{i+1})}{\mathrm{Im}(E^{i-1} \rightarrow E^i)} = 0$.

ii) Using [Theorem 2.75], for each $i \geq 0$, we have

$$H^i_d(M, N) \cong H^i(L_d(M, E^\bullet)) \cong H^i(\mathrm{Hom}_R(M, L_d(E^\bullet)))$$
$$\cong H^i(\mathrm{Hom}_R(M, L_d(E^\bullet))) \cong \mathrm{Ext}^i_R(M, N).$$

Corollary 2.6. Let M be an R- module. Then the following statements hold:

i) If $L_d(M) = M$ then $D_d(M) = 0$. Moreover, for any R- module X and Y, $D_d(H^i_d(X)) = D_d(H^i_d(X, Y)) = 0$.

ii) $D_d(M) \cong D_d(M/L_d(M))$.

iii) $D_d(M) \cong D_d(D_d(M))$.

iv) $L_d(D_d(M)) = 0 = H_d^1(D_d(M))$.
v) $H_d^i(M) \cong H_d^i(D_d(M))$ for all $i \geq 2$.
Proof. i) By corollary 2.4, $H_d^1(M) = 0$. So, by lemma 2.2, the following sequence

$$0 \to L_d(M) \to M \to D_d(M) \to 0$$

is exact. Now, since $M = L_d(M)$, then $D_d(M) = 0$.

For the next part, it is enough to show that $L_d(H_d^i(X)) = H_d^i(X)$ and $L_d(H_d^i(X,Y)) = H_d^i(X,Y)$. Here, we show that only the second part. Clearly, $L_d(H_d^i(X,Y)) \subseteq H_d^i(X,Y)$. Now, let $x \in H_d^i(X,Y)$. So, conciderіng the definition of direct limit, there exists ideal $\mathfrak{a}$ of R with $\dim\frac{R}{\mathfrak{a}} \leq d$ and R- homomorphism $\varphi_{\mathfrak{a}}: \mathrm{Ext}_R^i(\frac{R}{\mathfrak{a}}, M) \to H_d^i(X,Y)$ such that $\varphi_{\mathfrak{a}}(y) = x$ for some $y \in \mathrm{Ext}_R^i(\frac{R}{\mathfrak{a}}, M)$. Thus, $\mathfrak{a}x = 0$ and so $x \in L_d(H_d^i(X,Y))$. Then $L_d(H_d^i(X,Y)) = H_d^i(X,Y)$.

ii) Using the following exact sequence and part i, the result is complete.

$$0 \to L_d(M) \to M \to \frac{M}{L_d(M)} \to 0$$

iii) Using lemma 2.2 and part i, the result is complet.
iv) By putting $D_d(M)$ instead of M in lemma 2.2 and using part iii, the result is complete.
v) Considering the exact sequence

$$0 \to L_d(M) \to M \to \frac{M}{L_d(M)} \to 0,$$

we get the following sequence

$$H_d^i(L_d(M)) \to H_d^i(M) \to H_d^i(\frac{M}{L_d(M)}) \to H_d^{i+1}(L_d(M)).$$

Using corollary 2.5 (i), $H_d^i(M) \cong H_d^i(\frac{M}{L_d(M)})$ for all $i \geq 1$.
Now, From the following exact sequence

$$0 \to \frac{M}{L_d(M)} \to D_d(M) \to H_d^1(M) \to 0,$$

The result is complete.

Theorem 2.7. Let M and N be two R- modules. Then

$$\left(R^i\left(D_d(M,-)\right)\right)_{i\geq 0} \cong \Big(\varinjlim_{\dim\frac{R}{\mathfrak{a}}\leq d} \mathrm{Ext}_R^i(\mathfrak{a}M,-)\Big)_{i\geq 0}$$

as connected sequence of functors.
Proof. Let $T = D_d(M,-)$ and $T^i = \varinjlim_{\dim\frac{R}{\mathfrak{a}}\leq d} \mathrm{Ext}_R(\mathfrak{a}M,-)$, for all $i \geq 0$.

Since direct limit is an exact functor, then $\{T^i\}$ is an strongly connected sequence of functors. But T^0 and T are naturally equivalent and for any injective module E, $T^i(E) = 0$ for all $i \geq 1$. Thus the result follows from [3, Theorem 1.3.5].

Corollary 2.8. Let M, N be two R-modules. If $D_d(M,-)$ Is exact functor then, $H_d^i(M,N) \cong \mathrm{Ext}_R^i(M,N)$ for all $i \geq 2$.

proof. Since $D_d(M,-)$ is exact functor, then $R^i(D_d(M,N)) = 0$ for all $i \geq 1$. Thus, using theorem 2.7, $\varinjlim_{\dim\frac{R}{\mathfrak{a}}\leq d} \mathrm{Ext}_R^i(\mathfrak{a}M,N) = 0$ for all $i \geq 1$.

From the short exact sequence

$$0 \to \mathfrak{a}M \to M \to \frac{M}{\mathfrak{a}M} \to 0,$$

we get the following exact sequence

$$\mathrm{Ext}_R^i(\mathfrak{a}M,N) \to \mathrm{Ext}_R^{i+1}(\frac{M}{\mathfrak{a}M},N) \to \mathrm{Ext}_R^{i+1}(M,N) \to \mathrm{Ext}_R^{i+1}(\mathfrak{a}M,N).$$

Therefore,

$$\varinjlim_{\dim\frac{R}{\mathfrak{a}}\leq d} \mathrm{Ext}_R^i(\mathfrak{a}M,N) \to \varinjlim_{\dim\frac{R}{\mathfrak{a}}\leq d} \mathrm{Ext}_R^{i+1}(\frac{M}{\mathfrak{a}M},N)$$

$$\to \varinjlim_{\dim\frac{R}{\mathfrak{a}}\leq d} \mathrm{Ext}_R^{i+1}(M,N) \to \varinjlim_{\dim\frac{R}{\mathfrak{a}}\leq d} \mathrm{Ext}_R^{i+1}(\mathfrak{a}M,N)$$

is exact sequence. Thus,

$$\varinjlim_{\dim\frac{R}{\mathfrak{a}}\leq d} \mathrm{Ext}_R^{i+1}\left(\frac{M}{\mathfrak{a}M},N\right) \cong \varinjlim_{\dim\frac{R}{\mathfrak{a}}\leq d} \mathrm{Ext}_R^{i+1}(M,N) \quad \text{for all } i \geq 1.$$

Then $H_d^i(M,N) \cong \mathrm{Ext}_R^i(M,N)$ for all $i \geq 2$.

Proposition 2.9. Let M and N be two R- modules. Then, there exists long exact sequence:

$$0 \to L_d(M,N) \to \mathrm{Hom}_R(M,N) \to D_d(M,N) \to H_d^1(M,N)$$
$$\to \mathrm{Ext}_R^1(M,N) \to R^1(D_d(M,N)) \to H_d^2(M,N) \to \dots$$

proof. Let a is an ideal of R with $\dim\frac{R}{\mathfrak{a}} \leq d$. From the short exact sequence

$$0 \to \mathfrak{a}M \to M \to \frac{M}{\mathfrak{a}M} \to 0$$

we get the following long exact sequence

$0 \to \mathrm{Hom}_R(\frac{M}{\mathfrak{a}M},N) \to \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(\mathfrak{a}M,N) \to \mathrm{Ext}_R^1(\frac{M}{\mathfrak{a}M},N) \to \dots$.

Now, by applying the functor $\varinjlim_{\dim\frac{R}{\mathfrak{a}}\leq d}$ and theorem 2.7, the result is complete.

Corollary 2.10. If M be projective R- module or N be injective R- module, then

$$0 \to L_d(M,N) \to \mathrm{Hom}_R(M,N) \to D_d(M,N) \to H_d^1(M,N) \to 0$$

Is exact sequence.

Proof. This is easy.

Corollary 2.11. Let R be a quotient of a catenary, biequidimensional ring and M be finitely generated R-module. If M be projective R- module or N be injective R- module, then for any $\mathfrak{p} \in \mathrm{Spec}(R)$

$$D_d(M,N)_{\mathfrak{p}} \cong D_{d-\dim\frac{R}{\mathfrak{p}}}(M_{\mathfrak{p}},N_{\mathfrak{p}}).$$

proof. Similar proof of corollaty 2.3 and also by [1, Lemma], the result is complete.

Corollary 2.12. Let M be a finitely generated R- module. then for any R- module N,

$0 \to L_d(M,N) \to \mathrm{Hom}_R(M,N) \to D_d(\mathrm{Hom}_R(M,N)) \to H_d^1(\mathrm{Hom}_R(M,N)) \to 0$

is exact sequence.

Proof. Since $L_d(M,N) \cong L_d(\mathrm{Hom}_R(M,N))$, thus using lemma 2.2, the result is complete.

Corollary 2.13. Let M and N be two R- modules such that $p = \mathrm{pd}_R(M)$. Then for all $i > p$,

$$R^{i-1}(D_d(M,N)) \cong H_d^i(M,N).$$

proof. Since $\mathrm{Ext}_R^i(M,N) = 0$ for any $i > p$, then using proposision 2.9, the result is complete.

□

Corollary 2.14. Let M and N be two R- modules such that M is finitely generated and $L_d(N) = N$. Then $D_d(M,N) = 0$. Moreover, for any R- module X and $i \geq 0$, $D_d(M,H_d^i(X)) = 0$.

proof. Using of corollary 2.5, proposition 2.9 and also

$L_d(M,N) \cong \mathrm{Hom}_R(M,N)$,

The claim is held. For the second part, it can be easily seen that

$L_d(H_d^i(X)) = H_d^i(X)$ for all $i \geq 0$.

Theorem 2.15. Let M and N be two R- modules. Then the following statements hold:

i) If M is finitely generated R- module, then

$$D_d(M,N) \cong D_d(M,D_d(N)) \cong D_d(M,\frac{N}{L_d(N)}).$$

ii) If $\mathrm{Ext}_R^1(M,N) = 0$, then

$D_d(D_d(M,N)) \cong D_d(\mathrm{Hom}_R(M,N))$ and also, for any $i > 1$,

$$H_d^i(D_d(M,N)) \cong H_d^i(\mathrm{Hom}_R(M,N)).$$

iii) If M is a flat R- module, then ,

$$D_d(D_d(M,N) \cong D_d(M,N),$$

And also
$L_d(D_d M, N)) \cong H_d^i(D_d(M,N)) = 0.$
Proof. i) Concidering the short exact sequence

$$0 \to L_d(M) \to M \to \frac{M}{L_d(M)} \to 0,$$

we get the following long exact sequence
$0 \to D_d(M, L_d(N)) \to D_d(M,N) \to D_d(M, \frac{N}{L_d(N)}) \to R^1(D_d(M, L_d(N)) \to \ldots .$
Using the corollry 2.14, $D_d(M, L_d(N)) = 0 = R^1(D_d(M, L_d(N))$. Thus,

$$D_d(M,N) \cong D_d\left(M, \frac{N}{L_d(N)}\right).$$

Now, from the short exact sequence

$$0 \to \frac{N}{L_d(N)} \to D_d(N) \to H_d^1(N) \to 0,$$

we obtain thelong exact sequence

$$0 \to D_d(M, \frac{N}{L_d(N)}) \to D_d(M, D_d(N)) \to D_d(M, H_d^1(N)) \to R^1(D_d(M, \frac{N}{L_d(N)})) \to \ldots .$$

But, by corollary 2.14, $D_d(M, H_d^1(N)) = 0$. Therefore the result is holds.
ii) Since $\mathrm{Ext}_R^1(M,N) = 0$, then from the proposition 2.9, we get the exact sequence

$$0 \to \frac{\mathrm{Hom}_R(M,N)}{L_d(M,N)} \to D_d(M,N) \to H_d^1(M,N) \to 0. \qquad (*)$$

Hence, the long sequence

$$0 \to D_d(\frac{\mathrm{Hom}_R(M,N)}{L_d(M,N)}) \to D_d(D_d(M,N)) \to D_d(H_d^1(M,N)) \to R^i(D_d(\frac{\mathrm{Hom}_R(M,N)}{L_d(M,N)})) \to \ldots$$

is exact. Now, by corollary 2.6 i), $D_d(\frac{\mathrm{Hom}_R(M,N)}{L_d(M,N)}) \cong D_d(D_d(M,N))$. But by [7, Theorem 2.75],

$$D_d(M,N) = \varinjlim_{\dim\frac{R}{\mathfrak{a}} \le d} \mathrm{Hom}_R(\frac{M}{\mathfrak{a}M}, N) \cong \varinjlim_{\dim\frac{R}{\mathfrak{a}} \le d} \mathrm{Hom}_R(\frac{R}{\mathfrak{a}} \otimes_R M, N)$$

$$\cong \varinjlim_{\dim\frac{R}{\mathfrak{a}} \le d} \mathrm{Hom}_R(\frac{R}{\mathfrak{a}}, \mathrm{Hom}_R(M,N)) = L_d(\mathrm{Hom}_R(M,N)).$$

Now, by corollary 2.6 ii), the claim first part is holds.
For the second part, from (*), we get the following long exact

$$\ldots \to H_d^{i-1}(H_d^1(M,N)) \to H_d^i(\frac{\mathrm{Hom}_R(M,N)}{L_d(M,N)}) \to H_d^i(D_d(M,N)) \to H_d^i(H_d^1(M,N)) \to \ldots .$$

Now, since $L_d(H_d^1(M,N)) = H_d^1(M,N))$, thus by corollary 2.5 i), the result is complete.
iii) Since M is a flat R- module, then by [3, Corollary3.59],

$$D_d(M,N) = \varinjlim_{\dim\frac{R}{\mathfrak{a}} \le d} \mathrm{Hom}_R(\mathfrak{a}M, N) \cong \mathrm{Hom}_R(\mathfrak{a} \otimes_R M, N)$$

$$\cong \varinjlim_{\dim\frac{R}{\mathfrak{a}} \le d} \mathrm{Hom}_R(\mathfrak{a}, \mathrm{Hom}_R(M,N)) = D_d(\mathrm{Hom}_R(M,N)).$$

Now, by corollary 2.6 iii), we have

$$D_d(D_d(M,N)) \cong D_d(D_d(\mathrm{Hom}_R(M,N))) \cong D_d(\mathrm{Hom}_R(M,N)) \cong D_d(M,N).$$

Now, by putting $D_d(M,N)$ instead of M in lemma 2.2, the second part holds.

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