ON functors $D_d(-)$ and $D_d(M,-)$

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1. Introduction

Throughout this paper, $R$ denotes a commutative Noetherian ring with unity and $d$ will be a non-negative integer. For $R$-modules $M$ and $N$, we define

1) $L_d(M) = \{ m \in M : \dim Rm \leq d \}$

2) $H^i_d(M) = \lim_{\dim R_a \leq d} \Ext^i_R \left( R_a, M \right)$ for all $i \geq 0$
3) \[ L_d(M, N) = \lim_{\dim_{R/a} \leq d} \Hom_R\left( \frac{R}{aM}, N \right) \]
4) \[ H^i_d(M, N) = \lim_{\dim_{R/a} \leq d} \Ext_R^i\left( \frac{M}{aM}, N \right) \quad \text{for all } i \geq 0 \]

It can be shown easily that \( L_d(-) \) and \( L_d(M, -) \) are additive covariant R-linear functors on the category of R-modules which are left exact, too. For R-modules M and N, we easily have\( H^i_d(M) \equiv R^i(L_d(M)) \) and \( H^i_d(M, N) \equiv R^i(L_d(M, N)) \) for all \( i \geq 0 \).

Using the results in [7, Theorem 2.75], it can be shown easily that
\[ L_d(M, N) \cong L_d\left( \Hom_R(M, N) \right) \]
Moreover, if M be finitely generated R-module, then by [5, Satz 3]
\[ L_d(M, N) \cong \Hom_R(M, L_d(N)) \]

2. Main results

the functors , in the section \( D_d(-) \) and \( D_d(M, -) \) where are introduced and the related theorems are proven M is an R-module.

**Definition 2.1.** For any R-module M and N, we define

i) \[ D_d(M) = \lim_{\dim_{R/a} \leq d} \Hom_R\left( a, M \right) \]

ii) \[ D_d(M, N) = \lim_{\dim_{R/a} \leq d} \Hom_R\left( aM, N \right) \]

\( D_d(-) \) and \( D_d(M, -) \) are additive covariant R-linear functors which are left exact too.

**Lemma 2.2.** Let M be an R-module. Then the following sequence is exact:
\[ 0 \to L_d(M) \to M \to D_d(M) \to H^1_d(M) \to 0 \]
also, for any \( i \geq 1 \), \( R^i(D_d(M)) \cong H^{i+1}_d(M) \).

**proof:** Let \( a \) and \( b \) be two ideals of \( R \) such that \( a \leq b \), \( \dim_{R/a} \leq d \) and \( \dim_{R/b} \leq d \).

So, there is the following commutative diagram of R-modules and R-homomorphisms with exact rows:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & a & \longrightarrow & R & \longrightarrow & \frac{R}{a} & \longrightarrow & 0 \\
& & | & & | & & | & & |
0 & \longrightarrow & b & \longrightarrow & R & \longrightarrow & \frac{R}{a} & \longrightarrow & 0 \\
\end{array}
\]

Thus, we get the following commutative diagrams of R-modules and R-homomorphisms with exact rows:

\[
\begin{array}{cccccccccc}
0 & \longrightarrow & \Hom_R\left( \frac{R}{b}, M \right) & \longrightarrow & \Hom_R\left( R, M \right) & \longrightarrow & \Hom_R\left( b, M \right) & \longrightarrow & \Ext_R^1\left( \frac{R}{b}, M \right) & \longrightarrow & 0 \\
& & | & & | & & | & & | & & |
0 & \longrightarrow & \Hom_R\left( \frac{R}{a}, M \right) & \longrightarrow & \Hom_R\left( R, M \right) & \longrightarrow & \Hom_R\left( a, M \right) & \longrightarrow & \Ext_R^1\left( \frac{R}{a}, M \right) & \longrightarrow & 0 \\
\end{array}
\]

and for any \( i \geq 1 \),
Now, by applying direct limit in above commutative diagrams, the result follows.

**Corollary 2.3.** If \( R \) is the quotient of a catenary, biequidimensional ring, then for any \( p \in \text{Spec}(R) \)
\[ D_a(M)_p = D_{d - \dim^R_{a}(M)_p} \]

Proof. Using [1, Lemma], \( H^1_a(M)_p \equiv H^1_{d - \dim^R_{a}(M)_p} \) for all \( i \geq 0 \). Now, by commutative diagram with exact sequence
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & L_d(M)_p & \longrightarrow & M_p & \longrightarrow & D_d(M)_p & \longrightarrow & H^1_d(M)_p & \longrightarrow & 0 \\
& & \downarrow{\cong} & & \downarrow{id} & & \downarrow{\cong} & & \downarrow{\cong} & \\
0 & \longrightarrow & L_d(M)_{d - \dim^R_{a}(M)_p} & \longrightarrow & M_p & \longrightarrow & D_d(M)_{d - \dim^R_{a}(M)_p} & \longrightarrow & H^1_d(M)_{d - \dim^R_{a}(M)_p} & \longrightarrow & 0
\end{array}
\]
and five lemma, the proof is complete.

**Theorem 2.4.** If \( E \) is an injective \( R \)-module, then \( L_d(E) \) is also an injective \( R \)-module.

Proof. Let \( J \) be an ideal of \( R \) and let \( h: J \rightarrow L_d(E) \) be an \( R \)-homomorphism. We show that \( \exists x \in L_d(E) \) such that \( h(j) = jx \) for all \( j \notin J \). Since \( E \) is injective, \( \exists e \in E \) such that \( h(j) = je \) for all \( j \in J \). Now let \( J = \langle j_1, \ldots, j_n \rangle \). Thus for \( 1 \leq i \leq n \), there exists \( a_i \) such that \( \dim_{a_i}^R \leq d \) and \( a_i h(j_i) = 0 \). Put \( a = \prod_{i=1}^{n} a_i \).

Then \( \dim_{a}^R \leq d \) and \( ah(j) = 0 \). Since \( h(j) \leq Re \), using Artin-Rees lemma, \( \exists c \in N \) such that for all integers \( m \geq c \)
\[
a^m e \cap h(J) = a^{m-c}(e^a e \cap h(J)).
\]
Now for \( m = c + 1 \), we have \( a^{c+1} e \cap h(J) \subseteq ah(j) = 0 \).

Consequently the map \( \tilde{h}: a^{c+1} + J \rightarrow L_d(E) \) with \( \tilde{h}(r + s) = se \),
for all \( r \in a^{c+1} \) and \( s \in J \) is a homomorphism of \( R \)-module. Since \( E \) is injective, \( \exists x \in E \) such that \( \tilde{h}(r) = rx \) for all \( r \in a^{c+1} + J \). It is easy to see that for all \( r \in a^{c+1}, x \in L_d(E) \) and the proof is complete.

**Corollary 2.5.** Let \( M, N \) be two \( R \)-modules and let \( E^* \) be an injective resolution of \( N \). If \( M \) is finitely generated \( R \)-module and \( L_d(N) = N \), then
i) \( H^0_d(N) = 0 \) for all \( i \geq 1 \)
ii) \( H^i_d(M, N) \cong \text{Ext}^i_d(M, N) \) for all \( i \geq 0 \).

Proof. Using Theorem 2.2, we can construct an injective resolution
\[
0 \rightarrow N \rightarrow E^0 \rightarrow E^1 \rightarrow \ldots
\]
for \( N \) such that \( L_d(E^i) = E^i \) for all \( i \geq 0 \).

i) For each \( i \geq 1 \), \( H^i_d(N) = \ker \frac{\text{Ext}^i_d(M, E^1)}{\text{im}(\text{Ext}^i_d(M, E^0))} = 0 \).

ii) Using [Theorem 2.75], for each \( i \geq 0 \), we have
\[
H^i_d(M, N) \cong H^i(\text{Hom}_R(M, L_d(E^*)) \equiv H^i(\text{Hom}_R(M, L_d(E^*))) \cong \text{Ext}^i_d(M, N).
\]

**Corollary 2.6.** Let \( M \) be an \( R \)-module. Then the following statements hold:

i) If \( L_d(M) = M \) then \( D_d(M) = 0 \). Moreover, for any \( R \)-module \( X \) and \( Y \),
\[
D_d(H^0_d(X)) = D_d(H^0_d(X, Y)) = 0.
\]

ii) \( D_d(M) \cong D_d(M/L_d(M)) \).

iii) \( D_d(M) \cong D_d(D_d(M)) \).
iv) \( L_d(D_d(M)) = 0 = H^1_d(D_d(M)) \).

v) \( H^i_d(M) \cong H^i_d(D_d(M)) \) for all \( i \geq 2 \).

Proof. i) By corollary 2.4, \( H^1_d(M) = 0 \). So, by lemma 2.2, the following sequence

\[
0 \rightarrow L_d(M) \rightarrow M \rightarrow D_d(M) \rightarrow 0
\]

is exact. Now, since \( M = L_d(M) \), then \( D_d(M) = 0 \).

For the next part, it is enough to show that \( L_d(H^i_d(X)) = H^i_d(X) \) and \( L_d(H^i_d(X,Y)) = H^i_d(X,Y) \). Here, we show that only the second part. Clearly, \( L_d(H^i_d(X,Y)) \subseteq H^i_d(X,Y) \). Now, let \( x \in H^i_d(X,Y) \). So, considering the definition of direct limit, there exists ideal \( a \) of \( R \) with \( \dim_{R_a}^R \leq d \) and \( R \)-homomorphism \( \varphi_a: \Ext^i_R(R_a, M) \rightarrow H^i_d(X,Y) \) such that \( \varphi_a(y) = x \) for some \( y \in \Ext^i_R(R_a, M) \). Thus, \( ax = 0 \) and so \( x \in L_d(H^i_d(X,Y)) \). Then \( L_d(H^i_d(X,Y)) = H^i_d(X,Y) \).

ii) Using the following exact sequence and part i, the result is complete.

\[
0 \rightarrow L_d(M) \rightarrow M \rightarrow \frac{M}{L_d(M)} \rightarrow 0
\]

iii) Using lemma 2.2 and part i, the result is complete.

iv) By putting \( D_d(M) \) instead of \( M \) in lemma 2.2 and using part iii, the result is complete.

v) Considering the exact sequence

\[
0 \rightarrow L_d(M) \rightarrow M \rightarrow \frac{M}{L_d(M)} \rightarrow 0,
\]

we get the following sequence

\[
H^i_d(L_d(M)) \rightarrow H^i_d(M) \rightarrow H^i_d\left(\frac{M}{L_d(M)}\right) \rightarrow H^{i+1}_d(L_d(M)).
\]

Using corollary 2.5 (i), \( H^i_d(M) \cong H^i_d\left(\frac{M}{L_d(M)}\right) \) for all \( i \geq 1 \).

Now, from the following exact sequence

\[
0 \rightarrow \frac{M}{L_d(M)} \rightarrow D_d(M) \rightarrow H^i_d(M) \rightarrow 0,
\]

The result is complete.

**Theorem 2.7.** Let \( M \) and \( N \) be two \( R \)-modules. Then

\[
\left( R^i\left( D_d(M, -) \right) \right)_{i \geq 0} \cong \varprojlim_{\dim_{R_a}^R \leq d} \Ext^i_R(aM, -)_{i \geq 0}
\]

as connected sequence of functors.

Proof. Let \( T = D_d(M, -) \) and \( T^i = \varprojlim_{\dim_{R_a}^R \leq d} \Ext^i_R(aM, -) \), for all \( i \geq 0 \).

Since direct limit is an exact functor, then \( (T^i) \) is an strongly connected sequence of functors. But \( T^0 \) and \( T \) are naturally equivalent and for any injective module \( E \), \( T^i(E) = 0 \) for all \( i \geq 1 \). Thus the result follows from [3, Theorem 1.3.5].

**Corollary 2.8.** Let \( M, N \) be two \( R \)-modules. If \( D_d(M, -) \) is exact functor then \( H^i_d(M, N) \cong \Ext^i_R(M, N) \) for all \( i \geq 2 \).

Proof. Since \( D_d(M, -) \) is exact functor, then \( R^i(D_d(M, N)) = 0 \) for all \( i \geq 1 \). Thus, using theorem 2.7, \( \varprojlim_{\dim_{R_a}^R \leq d} \Ext^i_R(aM, N) = 0 \) for all \( i \geq 1 \).

From the short exact sequence

\[
0 \rightarrow aM \rightarrow M \rightarrow \frac{M}{aM} \rightarrow 0,
\]

we get the following exact sequence

\[
\Ext^i_R(aM, N) \rightarrow \Ext^{i+1}_R\left(\frac{M}{aM}, N\right) \rightarrow \Ext^{i+1}_R(M, N) \rightarrow \Ext^{i+1}_R(aM, N).
\]

Therefore,

\[
\varprojlim_{\dim_{R_a}^R \leq d} \Ext^i_R(aM, N) \rightarrow \varprojlim_{\dim_{R_a}^R \leq d} \Ext^{i+1}_R\left(\frac{M}{aM}, N\right) \rightarrow \varprojlim_{\dim_{R_a}^R \leq d} \Ext^{i+1}_R(M, N) \rightarrow \varprojlim_{\dim_{R_a}^R \leq d} \Ext^{i+1}_R(aM, N).
\]
\[ \lim_{\dim R \leq d} \text{Ext}_R^{i+1}(M, N) \rightarrow \lim_{\dim R \leq d} \text{Ext}_R^{i+1}(\alpha M, N) \]

is exact sequence. Thus,

\[ \lim_{\dim R \leq d} \text{Ext}_R^{i+1} \left( \frac{M}{\alpha M}, N \right) \equiv \lim_{\dim R \leq d} \text{Ext}_R^{i+1}(M, N) \quad \text{for all } i \geq 1. \]

Then \( H_d^i(M, N) \equiv \text{Ext}_R^i(M, N) \) for all \( i \geq 2 \).

**Proposition 2.9.** Let \( M \) and \( N \) be two \( R \)-modules. Then, there exists long exact sequence:

\[ 0 \rightarrow L_d(M, N) \rightarrow \text{Hom}_R(M, N) \rightarrow D_d(M, N) \rightarrow H_d^1(M, N) \rightarrow \text{Ext}_R^1(M, N) \rightarrow \cdots \]

proof. Let \( a \) is an ideal of \( R \) with \( \dim a \leq d \). From the short exact sequence

\[ 0 \rightarrow \alpha M \rightarrow M \rightarrow \frac{M}{\alpha M} \rightarrow 0 \]

we get the following long exact sequence

\[ 0 \rightarrow \text{Hom}_R \left( \frac{M}{\alpha M}, N \right) \rightarrow \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(\alpha M, N) \rightarrow \text{Ext}_R^1 \left( \frac{M}{\alpha M}, N \right) \rightarrow \cdots \]

Now, by applying the functor \( \lim_{\dim R \leq d} \) and theorem 2.7, the result is complete.

**Corollary 2.10.** If \( M \) be projective \( R \)-module or \( N \) be injective \( R \)-module, then

\[ 0 \rightarrow L_d(M, N) \rightarrow \text{Hom}_R(M, N) \rightarrow D_d(M, N) \rightarrow H_d^1(M, N) \rightarrow 0 \]

is exact sequence.

Proof. This is easy.

**Corollary 2.11.** Let \( R \) be a quotient of a catenary, biequidimensional ring and \( M \) be finitely generated \( R \)-module. If \( M \) be projective \( R \)-module or \( N \) be injective \( R \)-module, then for any \( p \in \text{Spec}(R) \)

\[ D_d(M, N)_p \equiv D_d(\dim R, N)_p. \]

proof. Similar proof of corollary 2.3 and also by [1, Lemma], the result is complete.

**Corollary 2.12.** Let \( M \) be a finitely generated \( R \)-module, then for any \( R \)-module \( N \),

\[ 0 \rightarrow L_d(M, N) \rightarrow \text{Hom}_R(M, N) \rightarrow D_d(\text{Hom}_R(M, N)) \rightarrow H_d^1(\text{Hom}_R(M, N)) \rightarrow 0 \]

is exact sequence.

Proof. Since \( L_d(M, N) \equiv L_d(\text{Hom}_R(M, N)) \), thus using lemma 2.2, the result is complete.

**Corollary 2.13.** Let \( M \) and \( N \) be two \( R \)-modules such that \( p = \text{pd}_R(M) \). Then for all \( i > p \),

\[ R^{p-i}(D_d(M, N)) \equiv H_d^i(M, N). \]

Proof. Since \( \text{Ext}_R^i(M, N) = 0 \) for any \( i > p \), then using proposition 2.9, the result is complete.

**Corollary 2.14.** Let \( M \) and \( N \) be two \( R \)-modules such that \( M \) is finitely generated and \( L_d(N) = N \). Then \( D_d(M, N) = 0 \). Moreover, for any \( R \)-module \( X \) and \( i \geq 0 \), \( D_d(M, H_d^i(X)) = 0 \).

proof. Using of corollary 2.5, proposition 2.9 and also

\[ L_d(M, N) \equiv \text{Hom}_R(M, N), \]

The claim is held. For the second part, it can be easily seen that

\[ L_d(H_d^i(X)) = H_d^i(X) \] for all \( i \geq 0 \).

**Theorem 2.15.** Let \( M \) and \( N \) be two \( R \)-modules. Then the following statements hold:

i) If \( M \) is finitely generated \( R \)-module, then

\[ D_d(M, N) \equiv D_d(M, D_d(N)) \equiv D_d(M, \frac{N}{L_d(N)}). \]

ii) If \( \text{Ext}_R^i(M, N) = 0 \), then

\[ D_d \left( D_d(M, N) \right) \equiv D_d(\text{Hom}_R(M, N)) \] and also, for any \( i > 1 \),

\[ H_d^i(D_d(M, N)) \equiv H_d^i(\text{Hom}_R(M, N)). \]

iii) If \( M \) is a flat \( R \)-module, then

\[ D_d(D_d(M, N)) \equiv D_d(M, N). \]
And also
\[ L_d(D_d(M, N)) \cong H^1_d(D_d(M, N)) = 0. \]
Proof. i) Considering the short exact sequence
\[ 0 \to L_d(M) \to M \to \frac{M}{L_d(M)} \to 0, \]
we get the following long exact sequence
\[ 0 \to D_d(M, L_d(N)) \to D_d(M, N) \to D_d(M, N) \cong \frac{N}{L_d(N)} \to R^1(D_d(M, L_d(N)) \to \ldots . \]
Using the corollary 2.14, \[ D_d(M, L_d(N)) = 0 = R^1(D_d(M, L_d(N))). \] Thus,
\[ D_d(M, N) \cong D_d\left( \frac{M}{L_d(N)} \right). \]
Now, from the short exact sequence
\[ 0 \to \frac{N}{L_d(N)} \to D_d(N) \to H^1_d(N) \to 0, \]
we obtain the long exact sequence
\[ 0 \to D_d(M, \frac{N}{L_d(N)}) \to D_d(D_d(M, N)) \to D_d(D_d(M, N)) \to R^1(D_d(M, \frac{N}{L_d(N)})) \to \ldots . \]
But, by corollary 2.14, \[ D_d(M, H^1_d(N)) = 0. \] Therefore the result is holds.
ii) Since \[ \text{Ext}^1_d(M, N) = 0, \] then from the proposition 2.9, we get the exact sequence
\[ 0 \to \text{Hom}_R(M, N) \to D_d(M, N) \to H^1_d(M, N) \to 0. \]
Hence, the long sequence
\[ 0 \to D_d\left( \frac{\text{Hom}_R(M, N)}{L_d(M, N)} \right) \to D_d(D_d(M, N)) \to D_d(H^1_d(M, N)) \to R^1(D_d\left( \frac{\text{Hom}_R(M, N)}{L_d(M, N)} \right)) \to \ldots . \]
is exact. Now, by corollary 2.6 i), \[ D_d\left( \frac{\text{Hom}_R(M, N)}{L_d(M, N)} \right) \cong D_d(D_d(M, N)). \] But by [7, Theorem 2.75],
\[ D_d(M, N) \cong \text{lim} \frac{R}{\text{Hom}_R(\mathfrak{a} M, N)} \cong \text{lim} \frac{R}{\text{Hom}_R(\mathfrak{a} \otimes_R M, N)} \]
\[ \cong \text{lim} \frac{R}{\text{Hom}_R(\mathfrak{a}, \text{Hom}_R(M, N))} = L_d(\text{Hom}_R(M, N)). \]
Now, by corollary 2.6 ii), the claim first part is holds.
For the second part, from (*), we get the following long exact
\[ \ldots \to H^1_d(\mathfrak{a}^i, \mathfrak{a}^j, N) \to H^1_d(M, N) \to H^1_d(M, N) \to \ldots . \]
Now, since \[ L_d(H^1_d(M, N)) = H^1_d(M, N), \] thus by corollary 2.5 i), the result is complete.
iii) Since \[ M \] is a flat \( R \)-module, then by [3, Corollary 3.59],
\[ D_d(M, N) \cong \text{lim} \frac{R}{\text{Hom}_R(\mathfrak{a} M, N)} \cong \text{Hom}_R(\mathfrak{a} \otimes_R M, N) \]
\[ \cong \text{lim} \frac{R}{\text{Hom}_R(\mathfrak{a}, \text{Hom}_R(M, N))} = D_d(\text{Hom}_R(M, N)). \]
Now, by corollary 2.6 iii), we have
\[ D_d(D_d(M, N)) \cong D_d(D_d(\text{Hom}_R(M, N))) \cong D_d(\text{Hom}_R(M, N)) \cong D_d(M, N). \]
Now, by putting \[ D_d(M, N) \] instead of \[ M \] in lemma 2.2, the second part holds.

References